Graph Realizations

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12 June 2009

Notation

Hypergraph

Denoted by $\mathscr{H}$, a graph in which generalized edges (called hyperedges) may connect more than two nodes [2]

- "For a set X and integer ℓ let us denote by $\binom{X}{\ell}$ the family of all subsets of X of size ℓ.
- Let $\mathscr{H} \subseteq 2^X$ be an arbitrary hypergraph over some finite base set X.
- Let furthermore ρ be a symmetric real function in two variables, ρ: $\mathbb{R}^2 \mapsto \mathbb{R}$, and
- let $k \in \mathbb{Z}_+$ be an integer threshold." [1]

Forming the Graph

"We shall associate to the triplet $(\mathscr{H},\rho,k)$ a graph $G^{\rho,k}_{\mathscr{H}}$ defined on the vertex set $V(G^{\rho,k}_{\mathscr{H}}) = \mathscr{H}$ and in which $(H,\ H') \in E(G^{\rho,k}_{\mathscr{H}})$ iff

$$\rho(\mid H \backslash H' \mid, \mid H' \backslash H \mid) \geq k,$$

where $H \backslash H' = \{x \in X \mid x \in H, x \notin H'\}$." [1]

Also consider $\rho \in \{\text{avg, max, min}\}$, where

- avg(a,b) $= \frac{a+b}{2}$
- max(a,b) $=$ max{a,b}
- min(a,b) $=$ min{a,b}

Graphs

Graph Realization

"Let us further say that a simple graph $G = (V, E)$ is $(\rho, k) - \textit{realizable}$ if $G = G^{\rho,k}_{\mathscr{H}}$ for some hypergraph $\mathscr{H}$ ...

Let us finally denote by $\mathscr{G}^{\rho,k}$ the family of (ρ, k)-realizable graphs." [1]

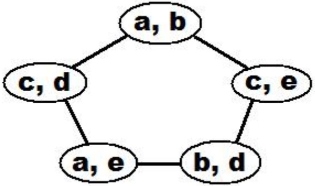

Figure: (min,2)- and (max,2)-realizable

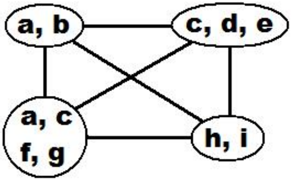

Figure: (avg,2)-realizable

Question

- An open question proposed by Dr. Endre Boros, Dr. Vladimir Gurvich, and Dr. Roy Meshulam in their paper Difference Graphs is whether it is possible to show that, given any set size ℓ where $\rho = \min$ and $k = 2$, all graphs can be realized.

Goal

- To begin this research, we are setting $\ell = 2$.
- We are also considering only connected graphs at this point.
- Therefore, we are looking for $\mathscr{G}^{min,2}$, where $\mathscr{H} \subseteq \binom{X}{2}$ and G is connected.
- For $\mathscr{H} \subseteq \binom{X}{2}$, $(H, H') \in E(G_{\mathscr{H}}) \Leftrightarrow \min(|\ H \backslash H'\ |, |\ H' \backslash H\ |) \geq 2$ $\Leftrightarrow H \cap H' = \emptyset$.

Goal

- To help with the goal, we are now looking for the minimal connected (min,2)-nonrealizable graphs, where $\mathscr{H} \subseteq \binom{X}{2}$.

Example

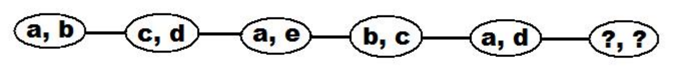

Figure: P_6 is nonrealizable since last vertex requires 3 elements

- The manner in which the graph was labeled does not lose generality. Thus, it will always be true that a P_6 is nonrealizable when $\ell = 2$.

Minimal Connected (min,2)-Nonrealizable Graphs

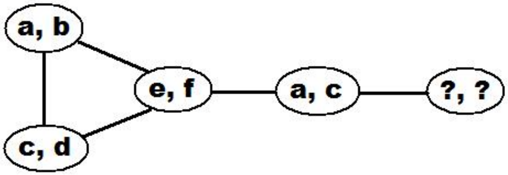

Figure: Need to use three elements in the unlabeled vertex

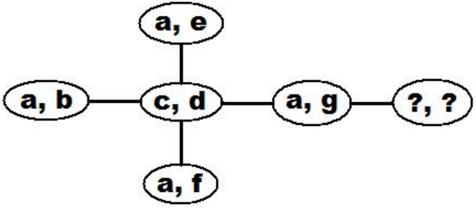

Figure: Need to use four elements in the unlabeled vertex

Acknowledgments

- Dr. Endre Boros
- Dr. Fred Roberts
- Dr. Gene Fiorini
- David Duncan
- DIMACS
- Rutgers University
- National Science Foundation (NSF)

References

- Boros, Endre, Vladimir Gurvich, and Roy Meshulam. "Difference Graphs." Discrete Mathematics. 276 (2004): 59-64.
- Weisstein, Eric W. "Hypergraph." *MathWorld*-A Wolfram Web Resource. 12 June 2009 <http://mathworld.wolfram.com/Hypergraph.html>.