

Multi-color graph pebbling

DIMACS REU Final Presentation

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DIMACS REU, Rutgers

July 17th, 2009

An introduction to graph pebbling

- Given a graph G , a set of vertices $V(G)$ and edges $E(G)$, a *pebbling distribution* is a placement of n pebbles on vertices of G .
- A legal *pebbling move* consists of removing two pebbles from a given vertex, and adding one to any adjacent vertex.
- The goal of pebbling is typically to designate a *root vertex*, or *target vertex*, and through a series of legal pebbling moves to place a pebble on the vertex.
- The *pebbling number* of the graph, $\pi(G)$, is the minimum number of pebbles such that for any pebbling distribution of $\pi(G)$ pebbles on G , any root vertex is reachable.
- A graph G is *Class 0* if its pebbling number is equal to $|V(G)|$.

Multi-color graph pebbling

We proposed a derivative problem, in which there are two possible colors for pebbles in a pebbling distribution, and a variety of goals for the game.

Notation and definitions

- For a simple graph G , we define a *bicolor pebble distribution* $Peb(r, g)$ as a distribution of pebbles in red and green along vertices of the graph. The initial distribution must be such that no vertex is occupied by pebbles of more than one color. For a vertex $v \in V(G)$, we can denote the number of pebbles of a given color distributed on v with $P_g(v)$ and $P_r(v)$.
- The edge set of a graph G is first partitioned into two sets $E_g(G)$ and $E_r(G)$, one of each color.
- A red-green pebbling game involves alternating legal pebbling moves for red and green pebbles - removing two pebbles of a given color from a single vertex and adding one of the same color to a vertex adjacent by an edge of that color.

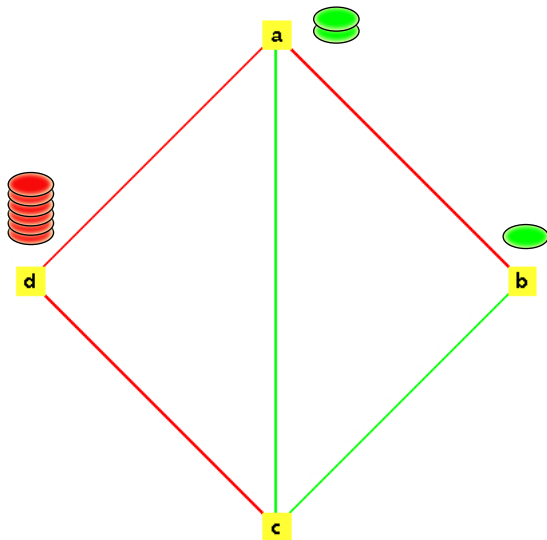
Notation and definitions

- A red pebble may not be placed on a vertex with two or more green pebbles, and vice versa.
- A vertex v with one pebble of each color forms a *block pair*, through which no other pebble may pass and from which none of the current occupants may move. One can subsequently consider the graph $G \setminus v$.

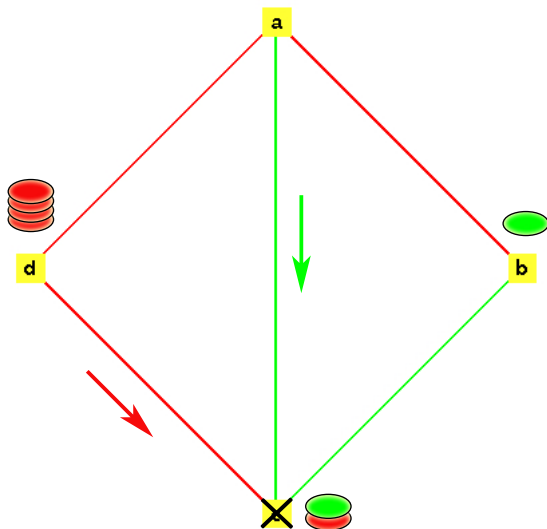
Game objectives?

- Two target vertices (one red, one green)
- One target vertex for one color, other color represents cooperative blocks.
- One target vertex for a block-pair
- Maximizing the number of simultaneously achievable block pairs.
- Competitive games with blocking as an objective, and unique target vertices for one or two colors.
- A modified game in which a legal pebbling move requires both a red and a green pebble, removing one and moving the other along a colored edge of the same color. In this case, a two-color pair represents a vehicle of motion rather than a block-pair.

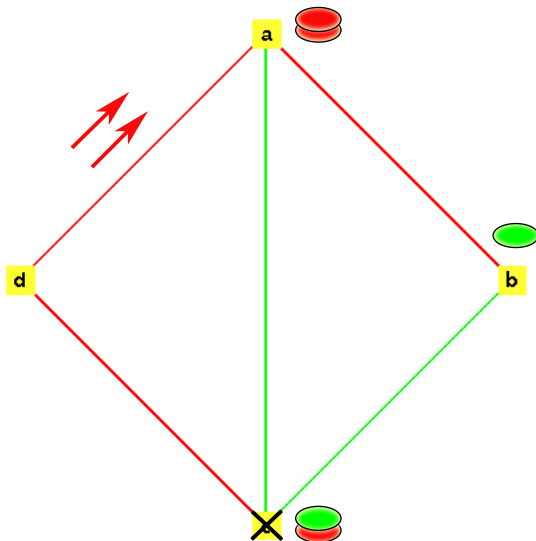
Example



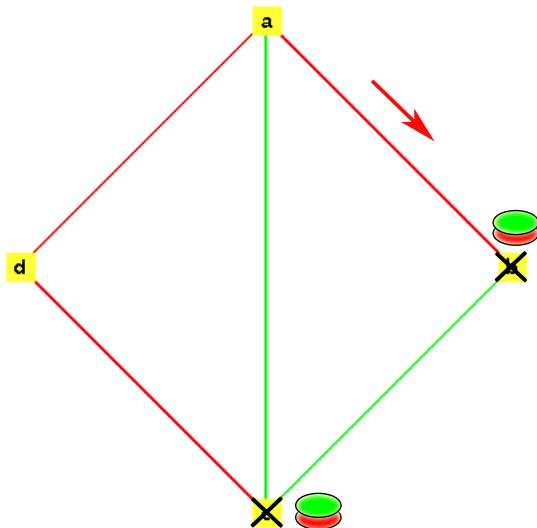
Example



Example



Example



Dual Class 0

An analagous definition to the regular graph pebbling problem would be to introduce *Dual Class 0 graphs*, graphs which can be edge-partitioned into two Class 0 graphs, those which can be decomposed into two complimentary subgraphs G and $H \setminus G$ such that $V(G) = V(H \setminus G) = V(H)$, and both G and $H \setminus G$ are Class 0.

Theorem

For $k \geq 9$, K_k is Dual Class 0.

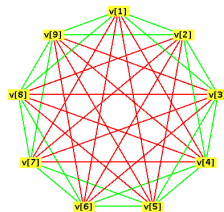
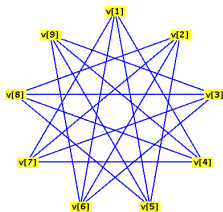
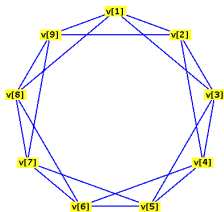
Families of Dual Class 0 graphs

Lemma

For $r = 2, 3, \dots$, $H = K_{4r+1}$ is Dual Class 0.

Proof.

We can illustrate this by construction, noting that a sufficient condition for Class 0 graph is that it is both 3-connected and of diameter 2. $\square$



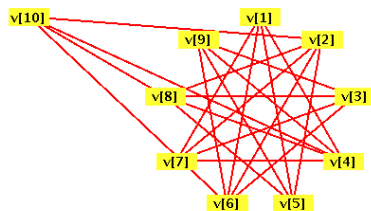
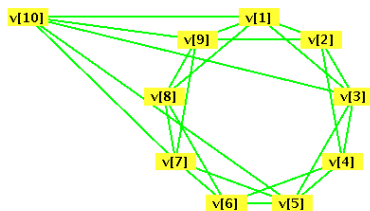
Families of Dual Class 0 graphs

Lemma

For $r = 2, 3, \dots$, $H = K_{4r+2}$ is Dual Class 0.

Proof.

Applying the results of our previous lemma, adding one vertex alternatively connected to every other vertex in G yields a still 3-connected and diameter two graph. □



Families of Dual Class 0 graphs

Theorem

For $k \geq 9$, K_k is Dual Class 0.

Proof.

The other cases follow by similar (but tedious to prove) construction. This will be left as an exercise. □

Theorem

For $k \geq 9$, the complete multipartite graph G with k partitions of equal size is Dual Class 0.

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We will utilize the results above to this end.

- First, label the partitions $i = 1, \dots, k$.

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- First, label the partitions $i = 1, \dots, k$.
- Next label all vertices v_{ij} , where i represents the partition it is contained in and j is an ordering within the partition.
- Slice the graph into n complete K_k subgraphs, and partition these as we have in the previous theorem for G and H/G .

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- Slice the graph into n complete K_k subgraphs, and partition these as we have in the previous theorem for G and H/G .
- Additionally, where a vertex is adjacent in G to an element of partition i_0 , we also add edges between that vertex and every other element of this partition.

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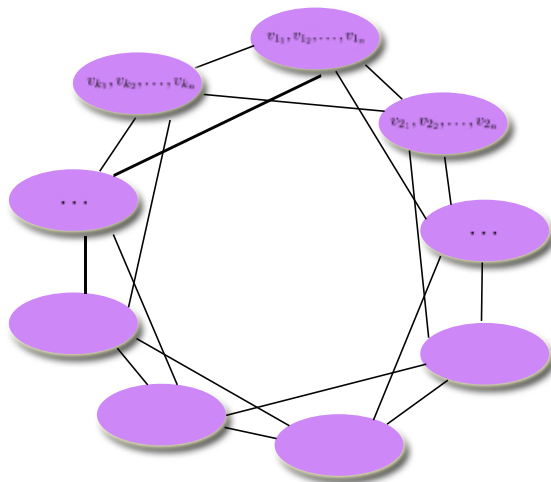
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- Next label all vertices v_{ij} , where i represents the partition it is contained in and j is an ordering within the partition.
- Slice the graph into n complete K_k subgraphs, and partition these as we have in the previous theorem for G and H/G .
- Additionally, where a vertex is adjacent in G to an element of partition i_0 , we also add edges between that vertex and every other element of this partition.
- We can show by induction that this graph is 3-connected, and the diameter must also be 2.



Crude drawing of multipartite construction



Analysis of the game

Analysis of the game

We've constructed a framework for our problem, and the next step is to proceed with strategy analysis.

Unsolvable distributions

- We say a pebble distribution $Peb(r, g)$ for a graph G is *unsolvable* if $\exists v \in V(G)$ s.t. $P_g(v) = P_r(v) = 0$, and the objective of the game cannot be achieved through any series of pebbling moves.
- We call a vertex v *r-blocked* if $P_g(v) = P_r(v) = 0$, and given the current distribution of red pebbles, for any distribution of these green pebbles, no series of consecutive green pebbling moves leads a green pebble to the vertex. The definition of *g-blocked* is analogous.
- Such a vertex is *permanently r-blocked* if no series of green or red moves will cause it to no longer be *r-blocked*.

An alternate problem

One distinguishable pebble

A similar problem, in which you have $n - 1$ indistinguishable 'red' pebbles and one distinguishable 'green' pebble. What is the minimum number of moves required to get the green pebble to a target unoccupied vertex?

Tentative results

This problem is more related to the original pebbling problem than the multi-color pebbling problem, as we can still look at something similar to the pebbling number, which we can refer to as $\pi_0(G)$. A naive lower bound for $\pi_0(G)$ is $n + 1$, and a naive upper bound is $\pi(G)2^{d(v,w)} + 2^{d(v,w)}$, where v is the starting vertex of the green pebble and w is the target vertex.

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- Isolate unsolvable conditions for arbitrary graphs
- Focus on a variant of the game which provides the most interesting results
- Develop necessary and sufficient conditions for the feasibility of a solution
- Characterize the optimization of a solution.

Bibliography

- 1 Archer, A., "A Modern Treatment of the 15 Puzzle", American Math Monthly, Vol. 106, 793-799, 1999.
- 2 Clark, B., Milans, K., "The Complexity of Graph Pebbling", SIAM Journal on Discrete Mathematics, Vol. 20, Iss. 3, 769-798, 2006.
- 3 Fink, A., Guy, R., "Rick Wilson's Tricky Six puzzle", Mathematics Magazine, Vol. 82, No. 2, 83-102, April 2009.
- 4 Goral, G., Hassin, R., "Multi-Color Pebble Motion on Graphs", Algorithmica, February 2009.
- 5 Kornhauser, D., Miller, G.L., Spirakis, P., "Coordinating Pebble Motion on Graphs, The Diameter of Permutation Groups, and Applications", In 25th Annual Symposium on Foundations of Computer Science, pp. 241-250, Florida, October 1984.
- 6 Parberry, I., "A real-time algorithm for the $(n^2 - 1)$ -puzzle", Information Processing Letters, Vol. 56, Iss. 1, 23-28, October 1995.
- 7 Ratner, D., Warmuth, M., "Finding a Shortest Solution for the $N \times N$ extension of the 15-PUZZLE is Intractable", Proceedings of the 5th National Conference on AI, 168-172, 1986.
- 8 Ryan, M., "Graph Decomposition for Efficient Multi-robot Path Planning", in Proceedings of the 20th International Joint Conference on Artificial Intelligence, Hyderabad, India, pp. 2003-2008, IJCAI Conference, 2007.
- 9 Surynek, P., "Finding Sub-optimal Solutions for Problems of Path Planning for Multiple Robots in θ -like Environments", Technical Report, ITI Series, 2007-430, 15 pages, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic, 2009.
- 10 Surynek, P., "Path Planning for Multiple Robots in Bi-connected Environments", Technical Report, ITI Series, 2009-431, 17 pages, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic, 2009.
- 11 Surynek, P., "Towards Shorter Solutions for Problems of Path Planning for Multiple Robots in θ -like Environments", Proceedings of the 22nd International Florida Artificial Intelligence Research Society Conference (FLAIRS 2009), Sanibel Island, FL, USA, pp. 207-212, AAAI Press, 2009.
- 12 Wilson, R., "Graph Puzzles, Homotopy, and the Alternating Group", J. Combin. Theory Ser. B., 86-96, 1974.