

Investigation of Properties of the Θ_0 Graph

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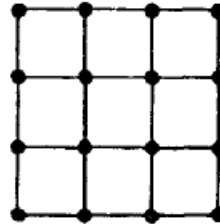
The 15-Puzzle

- Famous puzzle which rose to fame at the end of the 19th century.
- Problem posed to public by Noyes Chapman: is it possible, using only transpositions of the blank spot and adjacent squares, to interchange the positions of blocks 14 and 15 while returning all other blocks to their original positions?
 - In short, no; any chain of moves which returns the blank square to the lower right corner does so in an even number of moves. But the proposed scenario requires an odd permutation, which is impossible.
- The 15-puzzle in and of itself is a well documented problem.



Notable Results

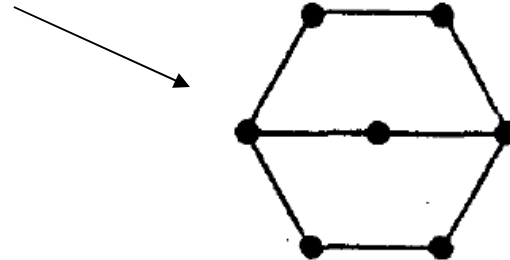
- When abstracted to a graph theoretical scenario in which the $n+1$ vertices of a graph G represent the cells, edges represent adjacencies, and labeling $\{1, 2, \dots, n, \emptyset\}$ denotes the n blocks, it has been shown that for most finite non-separable simple graphs, any permutation is reachable (Wilson, 1974).



- Consider a derivative graph of this problem, where possible labelings of G are the vertices, and edges connect two labelings which directly result from a legal move of the blank square.
- Every bipartite graph induces two connected components of reachable permutations that are equal in cardinality (the alternating group), and the reachable alignments of every non-exception form one connected component.

The 'Tricky Six' Graph

- The notable exception are bipartite graphs (as seen in the 15-puzzle), any polygon, and the graph θ_0 , shown here.



- This graph turns out to be very mathematically interesting. A hypothetical puzzle could be similar to the one proposed [here](#).
- The graph θ_0 bizarrely leads to six equivalence classes of reachable labelings.
- Recent research has noted similarities between this graph and a variety of other graphs and groups: the projective plane of order 4, the Hoffman-Singleton graph, the Steiner system $S(5,6,12)$, and the ternary Golay code C_{12} . (Fink, Guy, 2009).

Where to go next?

- We seek to generalize this problem in one of a number of ways:
 - We may investigate modifications to the model which should make the problem an easier one, for example considering the problem with $n-k$ blocks and k spaces, or adjusting adjacency relations.
 - We may investigate extremal properties of simple cases of graphs in the analogous optimization problem.
 - Or for example, an approach to a problem similar to [this](#).
 - We anticipate this will pare down the number of resulting equivalence classes, and could analyze the structures of these groups.

Bibliography

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- Alex Fink and Richard Guy, Rick Wilson's Tricky Six puzzle, *Mathematics Magazine*, Vol.82, No.2 (April 2009) 83-102.
 - Supplements to the article:
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- Jerry Slocum and Dic Sonneveld, The 15 Puzzle: How it Drove the World Crazy, *The Slocum Puzzle Foundation* (2006).
- Richard M. Wilson, Graph Puzzles, Homotopy, and the Alternating Group, *J. Combin. Theory Ser. B* (1974) 86-96.