On the Toric Varieties Associated with Bicolored Metric Trees

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Background in Algebraic Geometry

- An affine variety is a zero set of some polynomials in $C[x_1, \ldots, x_n]$.

- In general, we can replace C by an algebraically closed field K and define an affine variety to be a zero set of some polynomials in $K[x_1, \ldots, x_n]$. .

- Given an ideal I of $C[x_1, \ldots, x_n]$, denote by $V(I)$ the zero set of the polynomials in I. Then $V(I)$ is an affine variety.

Background in Algebraic Geometry

- Hilbert 's Lemma: For any proper ideal I in $C[x_1, \ldots, x_n]$, the corresponding variety $V(I)$ is nonempty
- Given a variety V in C^n, denote $I(V)$ to be the set of polynomials in $C[x_1, \ldots, x_n]$ which vanish in V.
- Define the quotient ring $C[V] = C[x_1, \ldots, x_n] / I(V)$ to be the coordinate ring of the variety V.
- From the Hilbert's Lemma, it follows that there is a natural bijective map from the set of maximal ideals of $C[V]$ to the points in the variety V.

Background in Algebraic Geometry

- We can write

$$Spec(C[V]) = V$$

where Spec($C[V]$) is the spectrum of the ring $C[V]$, i.e. the set of maximal ideals of $C[V]$.

- *Theorem:* $V = Spec(C[V]) = Hom(C[V], C)$

- Example: If $V = C^n$, then $I(V) = \{0\}$ and $C[V] = C[x_1, \ldots, x_n]$. Since the ideals $(x_1 - a_1, \ldots, x_n - a_n)$ are all the maximal ideals of $C[x_1, \ldots, x_n]$, we obtain

$$\text{Spec}(C[x_1, \ldots, x_n]) = C^n.$$

Background in Algebraic Geometry

- In C^n, we define the Zariski topology as follows: a set is closed if and only if it is an affine variety.

- Example: In C, the closed set in the Zariski topology is a finite set.

- Open sets in Zariski topology tend to be "large".

Background in Algebraic Geometry

- An irreducible variety V is a variety that can not be written as union of proper subvarieties $V = V_1 \cup V_2$.

- A variety V is irreducible if and only if its coordinate ring $C[V] = C[x_1, \ldots, x_n] / I(V)$ is prime.

Constructing Toric Varieties from Cones

- A convex cone σ in R^n is defined to be

$$\sigma = \{ r_1 v_1 + \ldots + r_k v_k \mid r_i \geq 0 \}$$

where $v_1, \ldots, v_n$ are given vectors in R^n

- Denote the standard dot product $< , >$ in R^n .

- Define the dual cone of σ to be the set of linear maps from R^n to R such that it is nonnegative in the cone σ.

$$\sigma^{\vee} = \{u \in R^n \mid <u , v> \geq 0 \text{ for all } v \in \sigma\}$$

Constructing Toric Varieties from Cones

- A cone σ is rational if its generators v_i are in Z^n.

- A cone σ is strongly convex if it doesn't contain a linear subspace

- From now on, all the cones σ we consider are rational and strongly convex.

- Denote $M = Hom(Z^n, Z)$, i.e. M contains all the linear maps from Z^n to Z

Constructing of Toric Varieties from Cones

- Gordon's Lemma: $\sigma^{\vee} \cap M$ is a finitely generated semigroup.

- Denote $S_\sigma = \sigma^{\vee} \cap M$. Then $C[S_\sigma]$ is a finitely generated commutative C – algebra.

- Define

$$U_\sigma = \mathrm{Spec}(C[S_\sigma])$$

Then U_σ is an affine variety.

Constructing Toric Varieties from Cones

- If $\tau \subset \sigma \subset R^n$ is a face, we have $C[S_\sigma] \subset C[S_\tau]$ and hence obtain a natural gluing map $U_\tau \to U_\sigma$. Thus all the U_τ fit together in U_σ.

Remarks

- Consider the face $\tau = \{0\} \subset \sigma$. Then $U_\tau = (C^*)^n = T_n$ here $C^* = C \setminus \{0\}$. Thus for every cone $\sigma \subset R^n$, there is an embedding $T_n \to U_\sigma$.

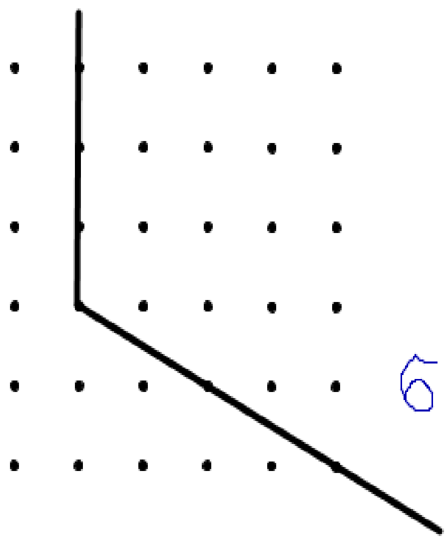
- There is a natural bijective correspondence between points in U_σ and elements in $Hom(S_\sigma , C)$.

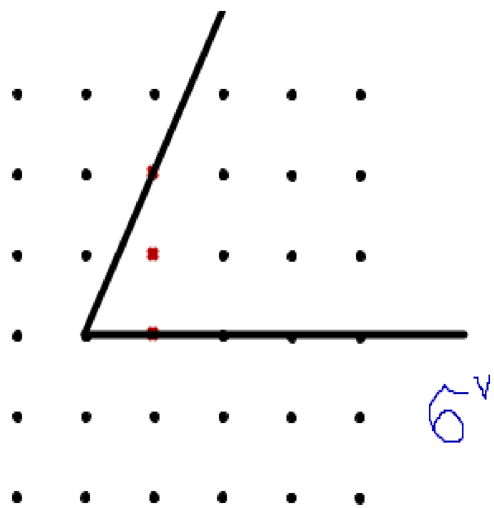
$$U_\sigma = Hom(S_\sigma , C).$$

Constructing Toric Varieties from Cones

Example: Consider σ to be generated by e_2 and $2e_1 - e_2$. Then the dual cone $\sigma^\vee$ is generated by e_1^* and $e_1^* + 2\, e_2^*$.

However, $S_\sigma = \sigma^\vee \cap M$ is generated by e_1^* , $e_1^* + e_2^*$ and $e_1^* + 2\, e_2^*$.





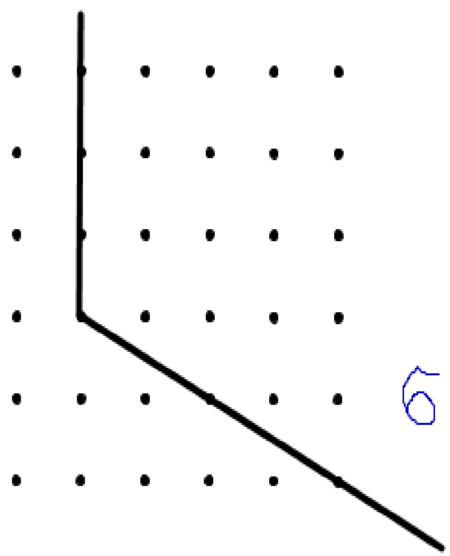
Constructing Toric Varieties from Cones

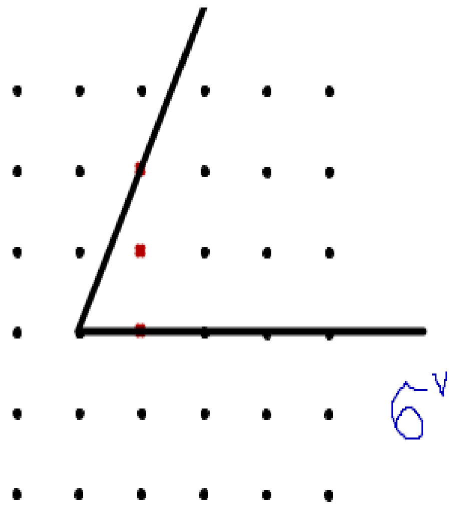
Let $x = e_1^{\ *}$, $xy = e_1^{\ *} + e_2^{\ *}$, $xy^2 = e_1^{\ *} + 2e_2^{\ *}$.

Hence $S_\sigma = \{x^a(xy)^b(xy^2)^c = x^{a+b+c}y^{b+2c} \mid a, b, c \in Z \geq 0\}$

Then $C[S_\sigma] = C[x, xy, xy^2] = C[u, v, w] / (v^2 - uw)$.

Thus $U_\sigma = \{ (u,v,w) \in C^3 \mid v^2 - uw = 0 \}$





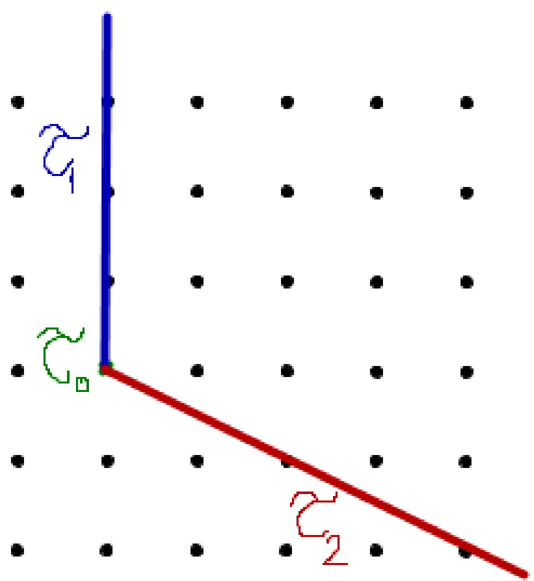
Constructing Toric Varieties from Cones

- If we denote τ_0, τ_1, τ_2 the faces of the cone σ. Then

$$U_{\tau^0} = (C^*)^2 \longrightarrow \{ (u,v,w) \in (C^*)^3 \mid v^2 - uw = 0 \}$$

$$U_{\tau^1} = (C^*) \times C \longrightarrow \{ (u,v,w) \in (C)^3 \mid v^2 - uw = 0, u \neq 0 \}$$

$$U_{\tau^2} = (C^*) \times C \longrightarrow \{ (u,v,w) \in (C)^3 \mid v^2 - uw = 0, w \neq 0 \}$$



Action of the Torus

- Recall that $V = Spec(C[V]) = Hom(C[V], C)$.

- From this, we can define the torus T_n action on U_σ as follows:
    1. $t \in T_n$ corresponds to $t^* \in Hom(S_{\{0\}}, C)$
    2. $x \in U_\sigma$ corresponds to $x^* \in Hom(S_\sigma, C)$
    3. Thus $t^* . x^* \in Hom(S_\sigma , C)$ and we denote $t . x$ the corresponding point in S_σ .
    4. We have the following group action:

$$\begin{aligned} T_n \times U_\sigma &\rightarrow U_\sigma \\ (t \;\; , x) &\mapsto t . x \end{aligned}$$

- Torus varieties: contains $T_n = (C^*)^n$ as a dense subset in Zariski topology and has a T_n action .

Orbits of the Torus Action

- The toric variety U_σ is a disjoint union of the orbits of the torus action T_n.

- Given a face $\tau \subset \sigma$. Denote by $\mathrm{N}_\tau \subset Z^n$ the lattice generated by τ.

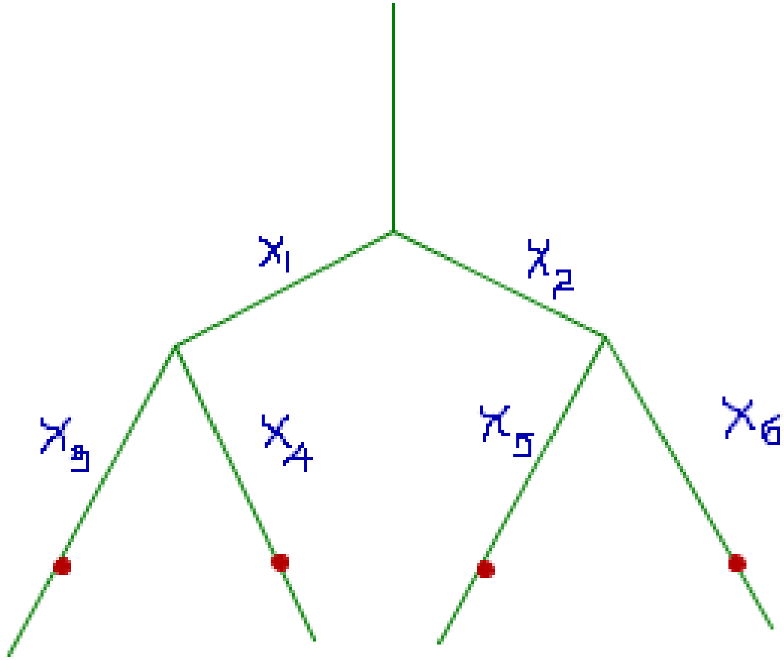
- Define the quotient lattice $\mathrm{N}(\tau) = Z^n / \mathrm{N}_\tau$ and let $O_\tau = T_{\mathrm{N}(\tau)}$, the torus of the lattice $\mathrm{N}(\tau)$. Hence $O_\tau = T_{\mathrm{N}(\tau)} = \cong (C^*)^k$

- There is a natural embedding of O_τ into an orbit of U_σ.

- Theorem There is a bijective correspondence between orbits and faces. The toric variety U_σ is a disjoint union of the orbits O_τ where τ are faces of the cones

Torus Varieties Associated With Bicolored Metric Trees

- Bicolored Metric Trees



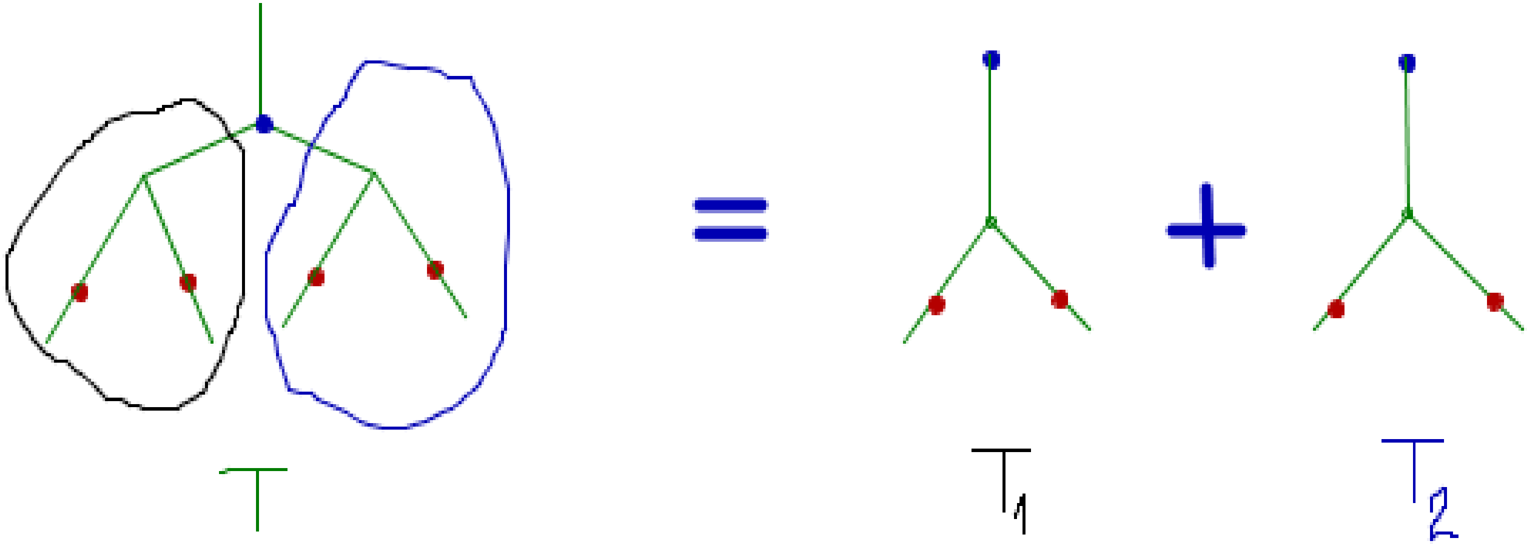
$$V(\mathrm{T}) = \{\ (x_1, x_2, x_3, x_4, x_5, x_6 \in C^6 \mid x_1 x_3 = x_2 x_5\,,\ x_3 = x_4\,,\ x_5 = x_6\}$$

Toric Varieties Associated With Bicolored Metric Trees

- Theorem *V*(T) is an affine toric variety.

Description of the Cones for the Toric Varieties

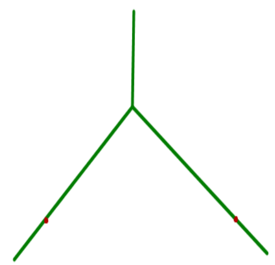
- Given a bicolored metric trees T.
- We can decompose the tree T into the sum of smaller bicolored metric trees $T_1, \ldots, T_m$.



Description of the Cones for the Toric Varieties

- We give a description of the cone $\sigma(T)$ by induction on the number of nodes n.

- For $n = 1$, we have



Thus $\sigma(T)$ is generated by e_1 in C.

- Suppose we constructed the generators for any trees T with the number of nodes less than n.

- Decompose the trees into smaller trees $T_1, \ldots, T_m$. For each tree T_k, denote by G_k the constructed set of generators of $\sigma(T_k)$

Description of the Cones for the Toric Varieties

- Then the set of generators of the cone $\sigma(T)$ is

$$G(T) = \{e_n + \alpha_1(z_1 - e_n) + \ldots + \alpha_m(z_m - e_n) \mid z_k \in G_i,\ \alpha_k \in \{0,1\}\}$$

Remarks

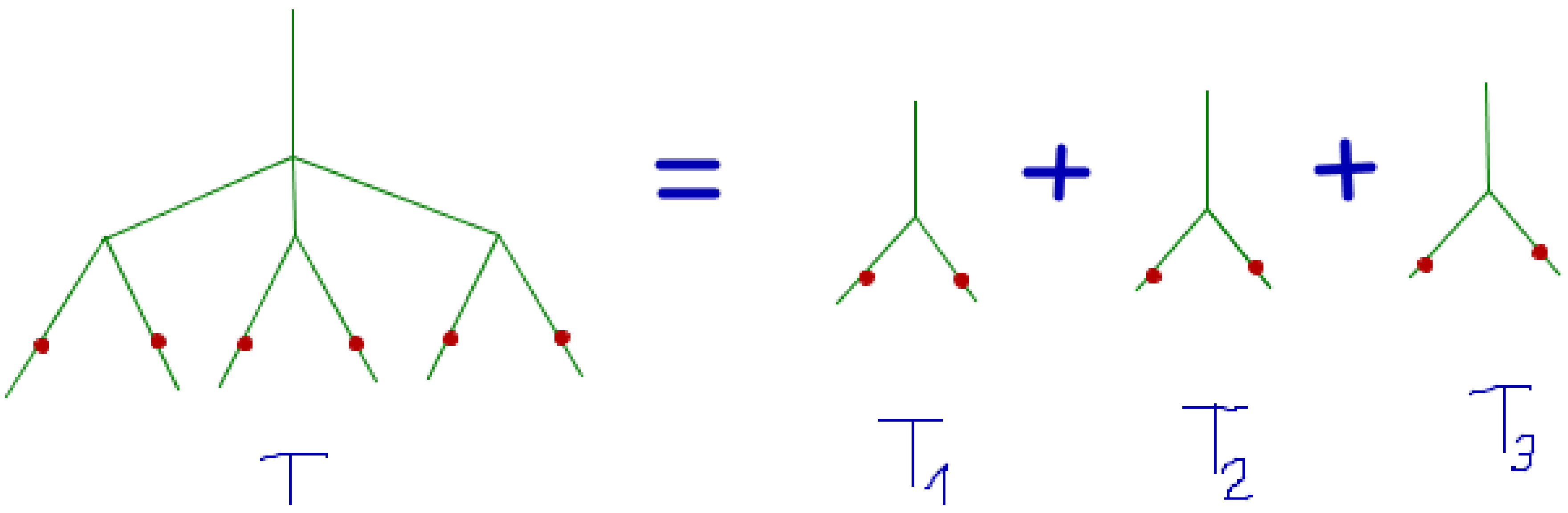
- The cone $\sigma(T)$ is n dimensional, where n is the number of nodes.
- The elements in G(T) corresponds bijectively to the rays (i.e. the 1-dimensional faces) of the cone $\sigma(T)$
- If the dimension of $\sigma(T_k)$ is n_k, then the dimension of $\sigma(T)$ is $(n_1+1)\ldots(n_m+1)$.

Description of the Cones for the Toric Varieties

Example: $T = T_1 + T_2 + T_3$

$$G(T_1) = \{e_1\},\ G(T_2) = \{e_2\},\ G(T_3) = \{e_3\}$$

$$G(T) = \{e_1, e_2, e_3, e_1 + e_2 - e_4, e_1 + e_3 - e_4, e_2 + e_3 - e_4, e_1 + e_2 + e_3 - 2e_4\}$$



Weil Divisors

- A prime Weil divisor D of a variety V is an irreducible subvariety of codimensional 1.
- A Weil divisor is an integral linear combination of prime Weil divisors.
- Given a bicolored metric tree T with variety V(T). We care about toric-invariant prime Weil divisors, i.e. $T_n(D) \subset D$.
- Theorem: $D = \mathrm{clos}(O_\tau)$ for some 1 dimensional face τ.
- There is a natural bijective correpondence between the set of prime Weil divisor and the set of generators G(T).

Weil Divisors

We can also list all the toric-invariant prime Weil divisors as follows:

- Let $x_1, \ldots, x_N$ be all the variables we label to the edges of T.

- We call a subset $Y = \{y_1, \ldots, y_m\}$ of $\{x_1, \ldots, x_N\}$ complete if it has the property that $x_k = 0$ if and only if $x_k \in Y$ when we set $y_1 = \ldots = y_m = 0$.

- A complete subset Y is called minimally complete if it doesn't contain any other complete subset.

Weil Divisors

- Lemma Y is a minimally complete subset if and only if the unique path from each colored point to the root contains exactly one edge with labelled variable y_k.

- Theorem D is a toric-invariant prime Weil divisor if and only if $D = \{ (x_1, \ldots, x_N) \in V(\mathrm{T}) \mid x_k = 0, \forall\, x_k \in Y\}$ for some minimally complete Y.

- Corollary There is a natural bijective correspondence between the set of toric-invariant prime Weil divisors and the set of minimally complete subsets.

Weil Divisors

We can describe the correpondence map in theorem by induction on the number of nodes n.

- For $n = 1$, it is simple.

- Decompose the tree T into $T_1, \ldots, T_m$ and let $x_1, \ldots, x_m$ be the labelled variables on the principal branches (which are now the roots of $T_1, \ldots, T_m$).

- Let D be a toric-invariant prime Weil divisor and Y be its corresponding minimally complete set. Let χ_Y be the character function on Y.

Weil Divisors

- If $x_k \notin Y$ for some $1 \le k \le m$, then D induces naturally a prime Weil divisor D_k of $V(\mathrm{T}_\mathrm{k})$, which by induction corresponds to an element z_k in $\mathrm{G}(\mathrm{T}_\mathrm{k})$.

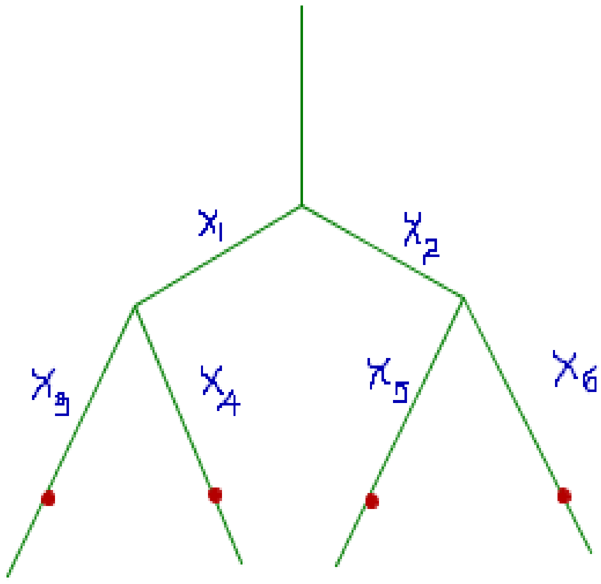
- If $x_k \in Y$ for some $1 \le k \le m$, define $z_k = 0$.

- D corresponds to $e_n + (1 - \chi_Y(x_1))\,(z_1 - e_n) + \ldots + (1 - \chi_Y(x_m))(z_m - e_n)$.

Weil Divisors

Example: There are 4 prime Weil divisors D_{12}, D_{156}, D_{234}, D_{3456} which corresponds to the minimal complete sets Y_{12}, Y_{156}, Y_{234}, Y_{3456}.

Decompose the tree into two trees T_1 and T_2. Then $G(T_1) = \{e_1\}$ and $G(T_2) = \{e_2\}$. Thus D_{12}, D_{156}, D_{234}, D_{3456} corresponds to e_3, e_2, e_1, $e_2 + e_3 - e_1$.



Cartier Divisors

- Given a variety V. For every point $p \in V$, we define the local ring $O_p = \{f(x)/g(x) \mid g(p) \neq 0, f, g$ are polynomials$\} \subset C(V)$.

- It is a theorem that if V is a normal affine variety, then O_p is a discrete valuation ring.

- Hence given any subvariety D of codimension 1 of V and $p \in D$, we can define $ord_{D,p}(f)$ for every function $f \in C(V)$.

- Informally speaking, $ord_{D,p}(f)$, determines the order of vanishing of f at p.

- It turns out that $ord_{D,p}$ doesn't depend on $p \in V$.

Cartier Divisors

- Theorem The toric variety $V(\mathrm{T})$ is normal for any tree T.

- It makes sense to talk about ord_D for each prime Weil divisor D (not necessary toric-invariant).

- For each f, define

$$div(f) = \sum_D (ord_D(f)).D$$

- If $ord_D(f) = 0$ for every non toric-invariant prime Weil divisor, we call $div(f)$ a toric-invariant Cartier divisors.

Cartier Divisors

- Theorem *div(f)* is toric-invariant if and only if *f* is a fraction of two monomials.

- Recall that we have constructed a natural correspondence between the prime Weil divisors of V(T) and the elements in G(T).

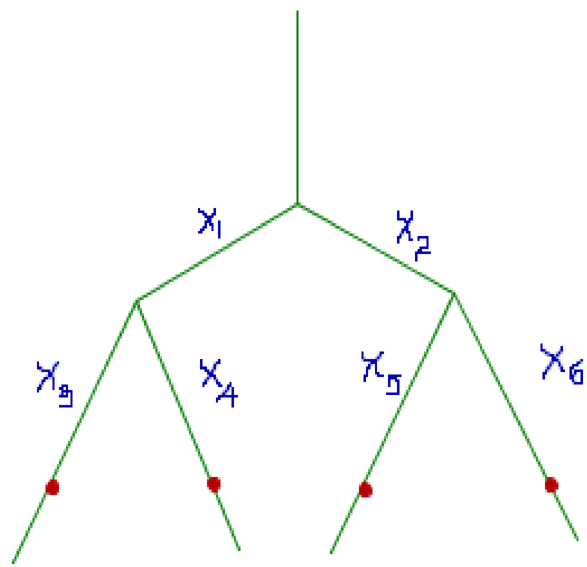
- Call $D_1, \ldots, D_N$ the prime Weil divisors of V(T) and $v_1, \ldots, v_N$ the corresponding elements in G(T).

- Recall that if the tree T has n nodes, then $v_1, \ldots, v_N$ are vectors in Z^n.

Cartier Divisors

- <u>Theorem</u> A Weil divisor $D = \sum a_i D_i$ is Cartier if and only if there exists a map $\varphi \in Hom(Z^n, Z)$ such that $\varphi(v_i) = a_i$.

Example The prime Weil divisors of T are D_{12}, D_{156}, D_{234}, D_{3456} corresponds to e_3, e_2, e_1, $e_2 + e_3 - e_1$. Hence $aD_{12} + bD_{156} + cD_{234} + dD_{3456}$ is Cartier if and only if $a + d = b + c$.



Cartier Divisors

- We can describe the generators of the set of Cartier divisors as follows:

1) Denote the nodes of T to be the point $P_1, \ldots, P_n$.

2) Call $Y_1, \ldots, Y_N$ the corresponding minimal complete set of the prime Weil divisors $D_1, \ldots, D_N$.

3) For each node P_k, let ϕ_k be a map defined on $\{D_1, \ldots, D_N\}$ such that $\phi_k(D_i) = 0$ if there is no element in Y_i in the subtree below P_k.

Otherwise $1 - \phi_k(D_i) =$ the number of branches of P_k that doesn't contain an element in Y_i.

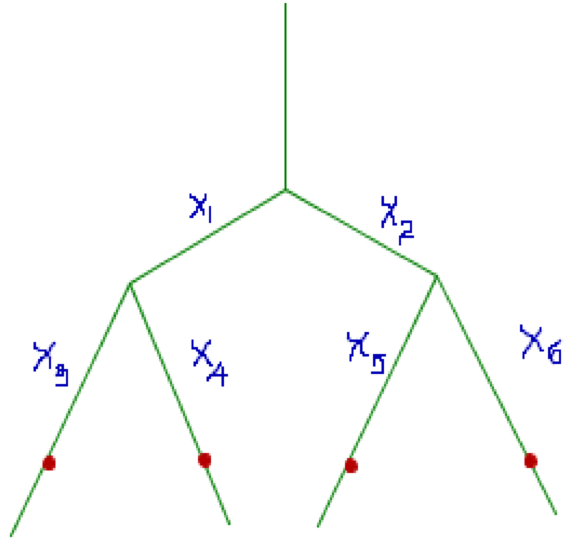
Cartier Divisors

- <u>Theorem</u> The set of Cartier divisors is generated by

$$\{\phi_k(D_1)\, D_1 + \ldots + \phi_k(D_N)\, D_N \mid 1 \le k \le m\}$$

Example: The generators of the Cartier divisors of T are

$D_{12} - D_{3456},\ D_{234} + D_{3456},\ D_{156} + D_{3456}$.



Summary of Results

Description of toric-invariant Weil divisors and Cartier divisors of the torus variety associated with a bicolored metric tree.

Possible future plan

Reference

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- Miles, R.: Undergraduate Algebraic Geometry, London Mathematical Society Student Texts. 12, Cambridge University Press 1988.
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