Periodic Orbits on a Billiard Table

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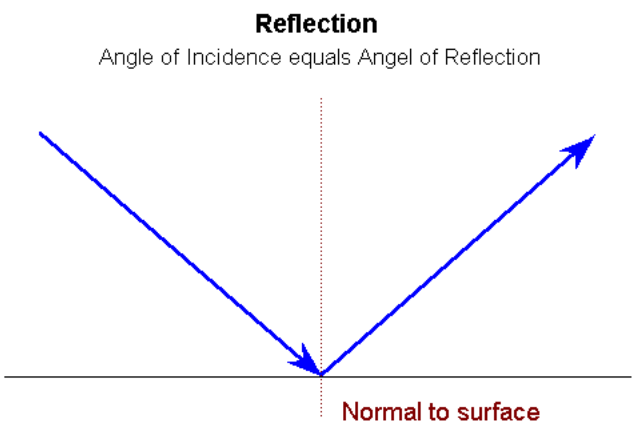
Think of an Air Hockey Table…

- Elastic collisions- angle of incidence is the angle of reflection.



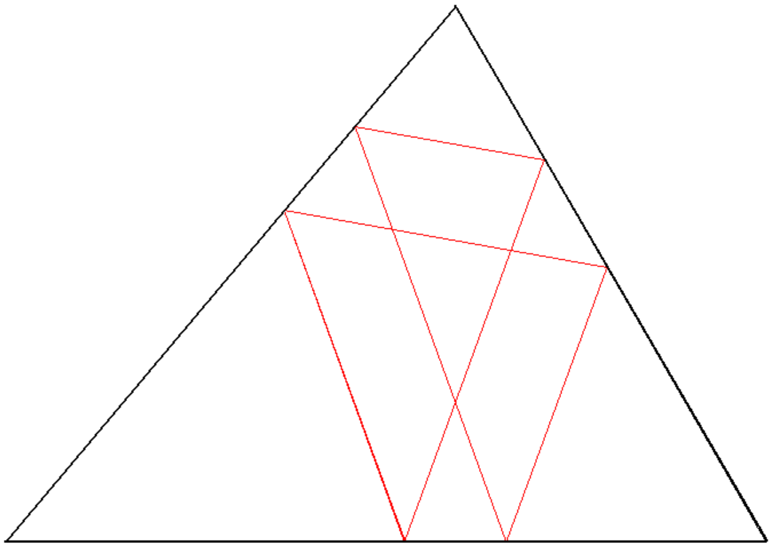
- Frictionless surface: puck continues indefinitely until a corner is struck.

Periodic Orbits

- We call the path of the puck its **orbit**.

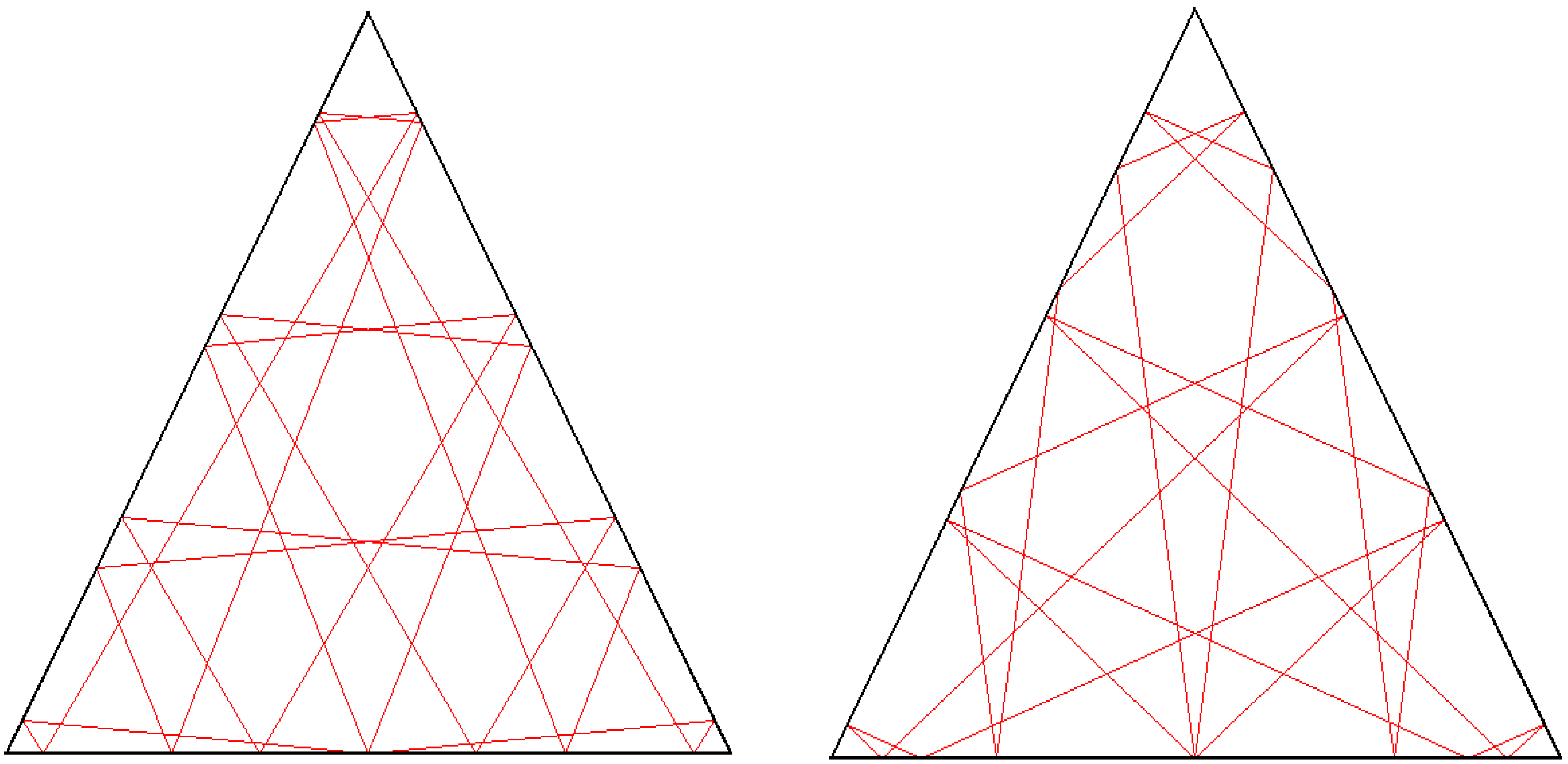
- Orbits which retrace themselves are called **periodic**.

- Periodic orbits that bounce *n* times before repeating have **period *n*.**



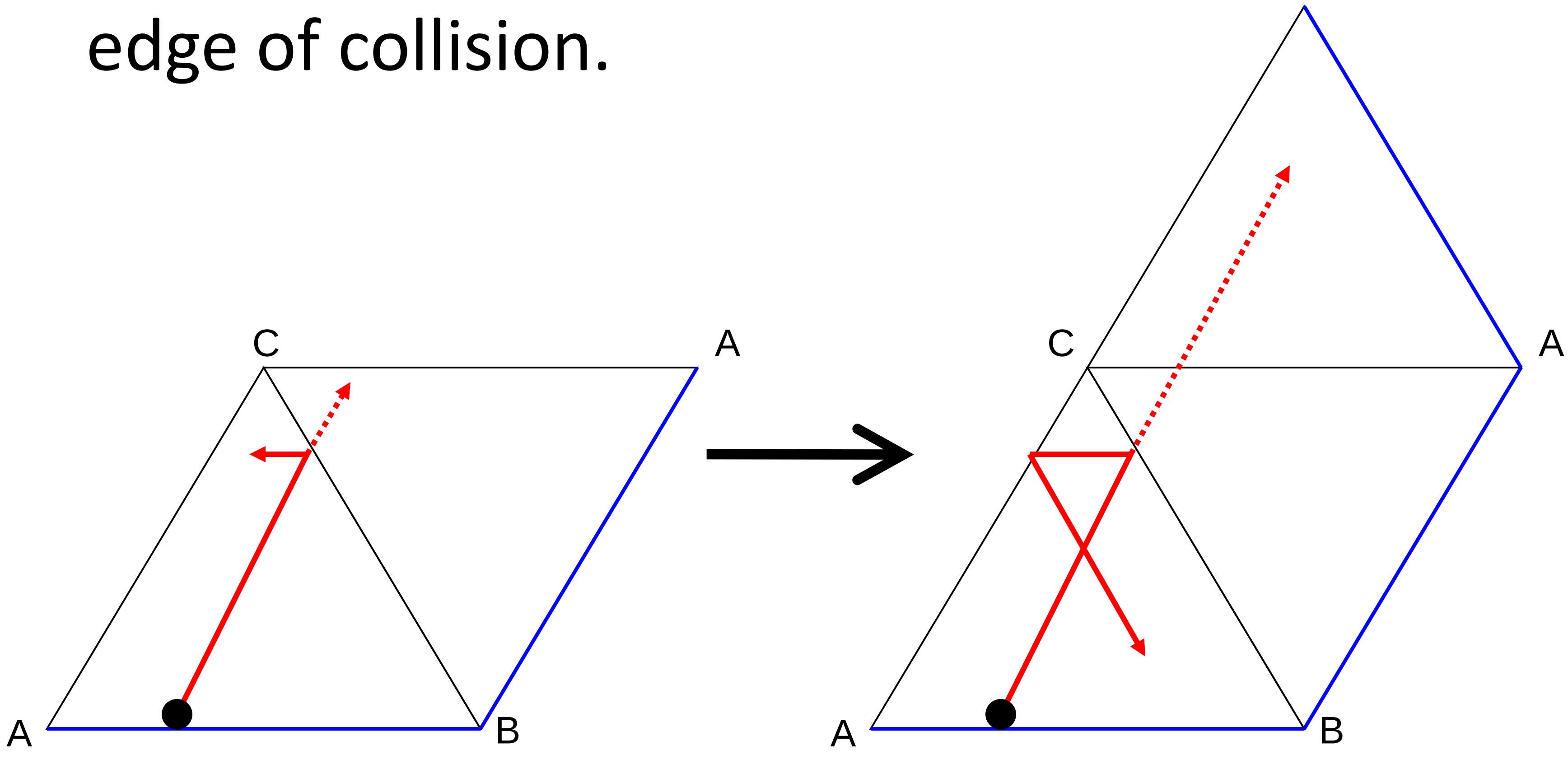
Problem: given a specific billiard table, count and classify all its [periodic orbit]s.

Examples:



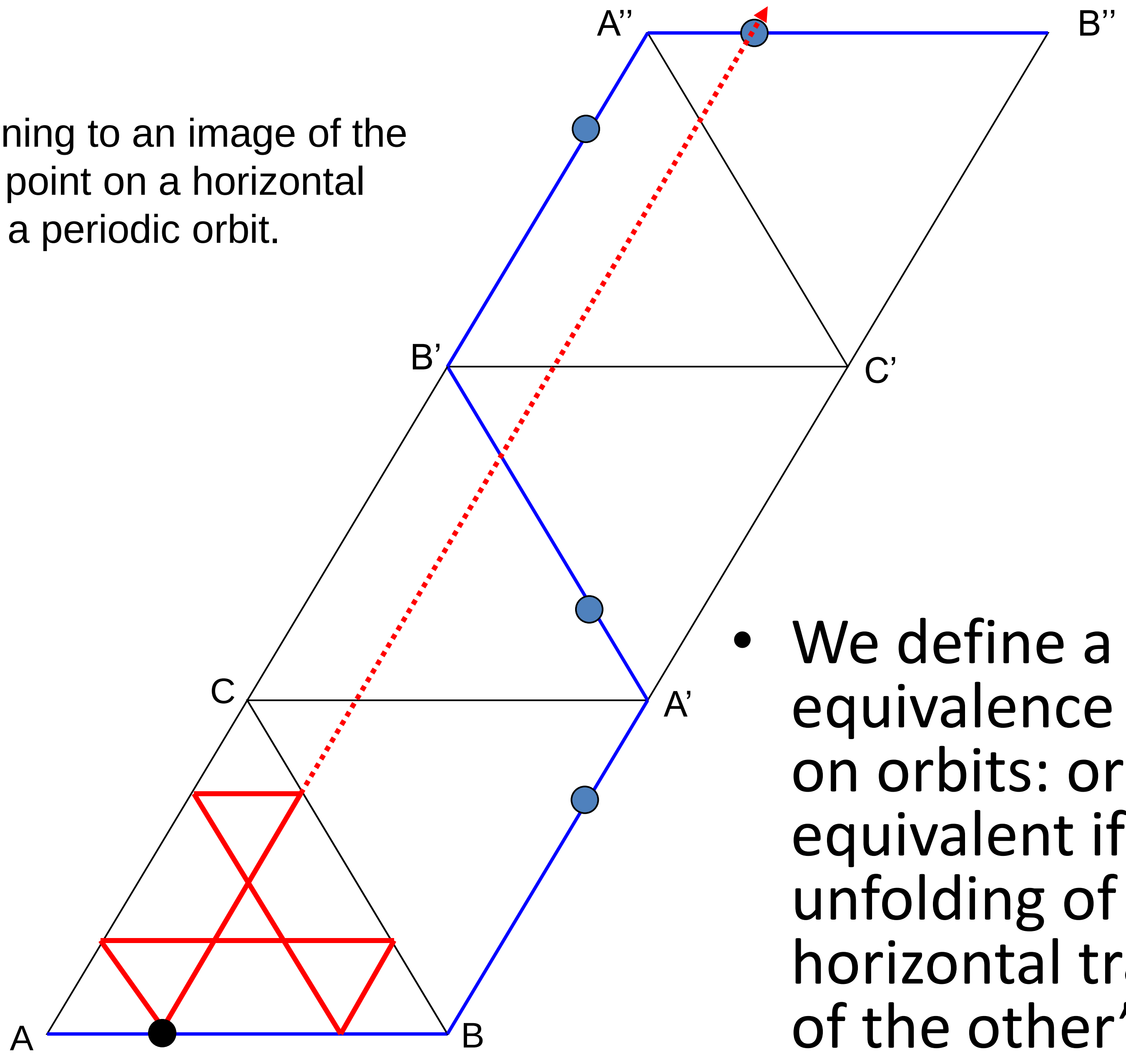
Unfolding

- We will use a technique from transformative geometry. At each bounce, reflect along the edge of collision.



- This reduces an orbit to a straight line in the plane.

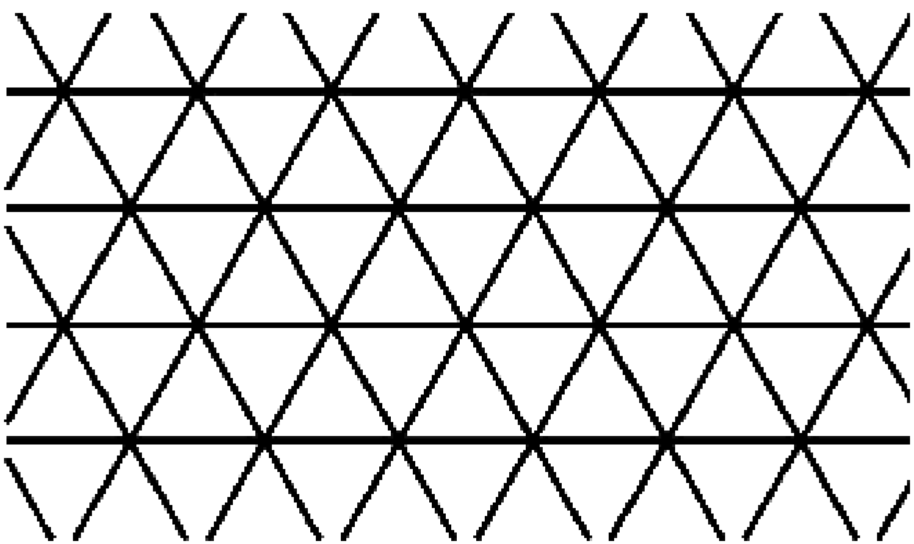
Returning to an image of the initial point on a horizontal gives a periodic orbit.



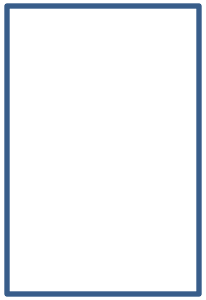
- We define a natural equivalence relation on orbits: orbits are equivalent if an unfolding of one is a horizontal translation of the other's.

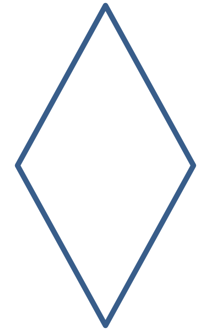
Unfolding gives… tessellations

- A tessellation is a tiling of the plane.
- Reflecting the equilateral triangle indefinitely across all edges in all directions gives a tiling.

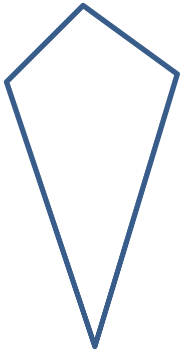


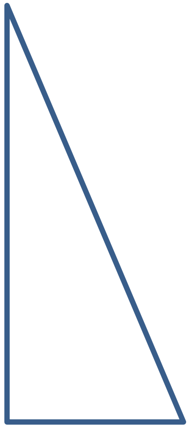
- We can exploit the tessellation's grid structure with a coordinate system.





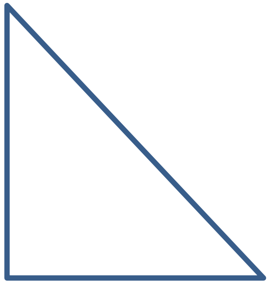
- Counting and classifying orbits on the equilateral triangle has been done previously.

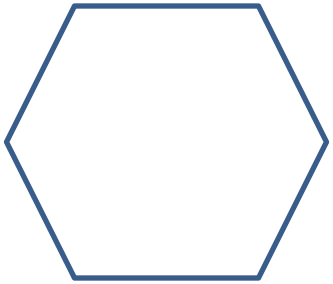


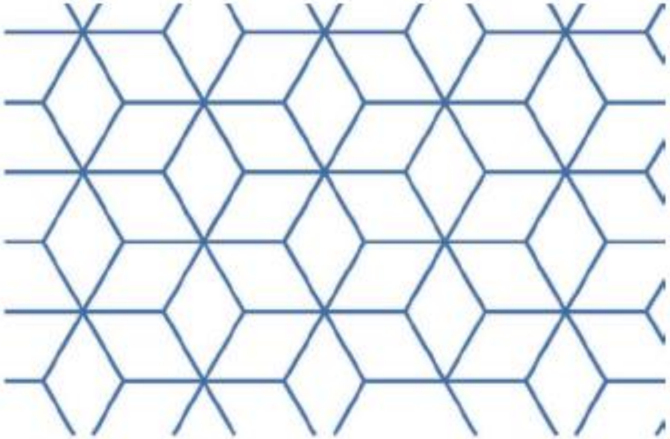


- There are 7 other polygons which tessellate the plane by edge reflections.

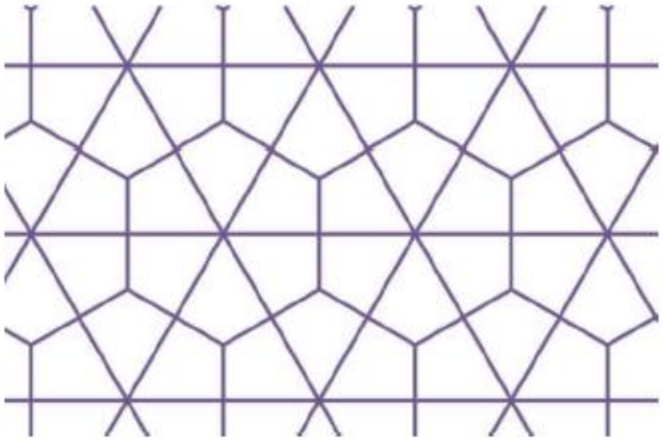
(regular hexagon; 60-90-120 kite; 120-rhombus; rectangle; 30-60-90, isosceles right, and 120-isosceles triangle)



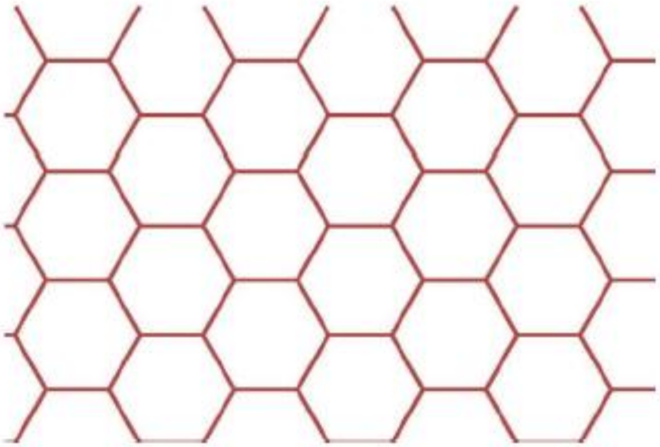




120 rhombus

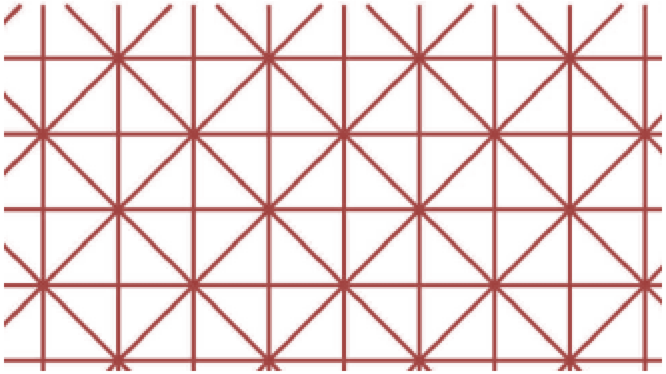


60-90-120 kite

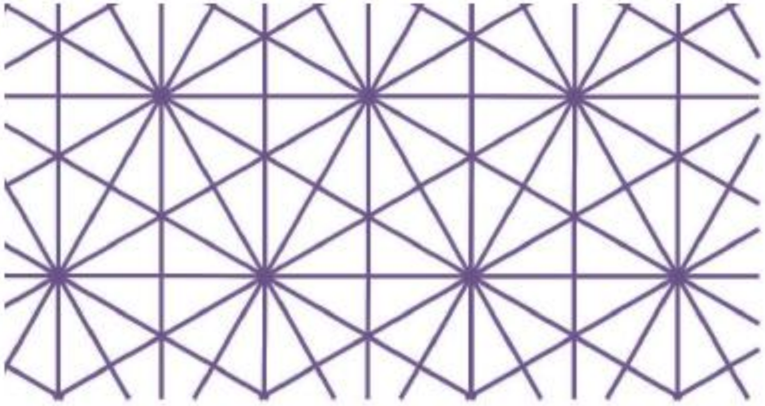


Regular hexagon

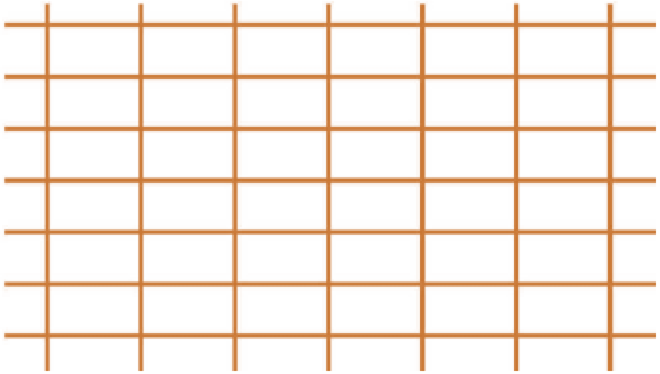
We hope to extend the unfolding/tessellation technique to these seven figures.



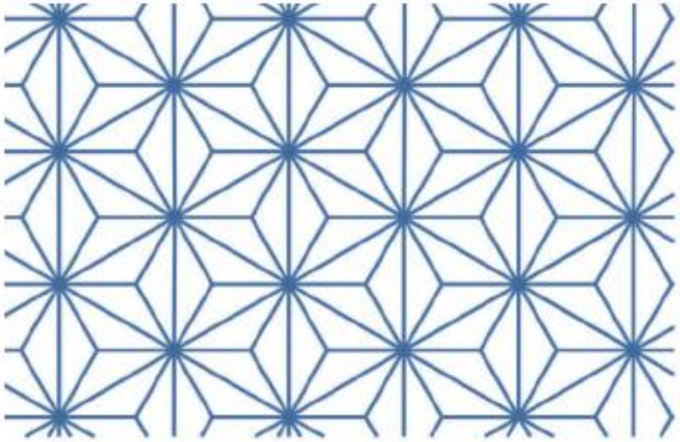
Isosceles right



30-60-90

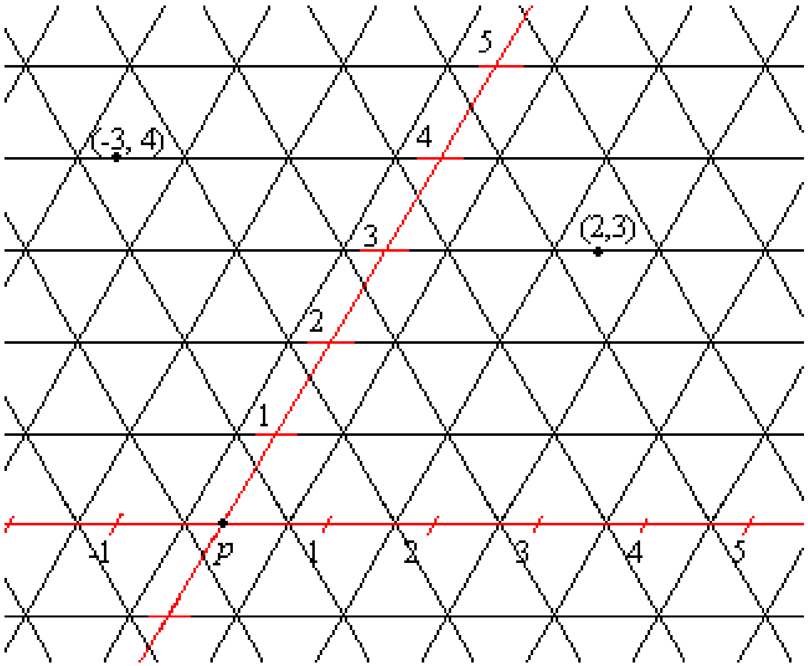


Rectangle



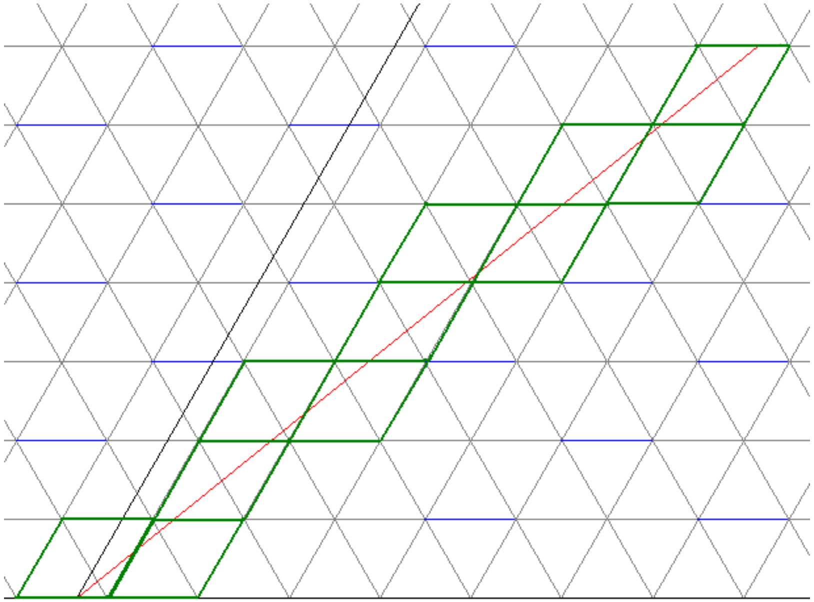
120 isosceles

Creating a Coordinate System



- Steps (equilateral triangle):
  - Make the y-axis a right-leaning (60 degree) diagonal line.
  - Choose a point at which the puck strikes the base as the origin.
- How can coordinate systems be constructed to study:
  - Hexagons?
  - Kites?
  - Rectangles?
  - Non-equilateral triangles?

Counting Orbits



The period 22 orbit (4, 7)

- Any periodic orbit unfolding will have unique (x,y) integer coordinates marking its ending point.

<u>Results on an equilateral triangle:</u>

- There is only one fundamental odd-period orbit (n=3) *(demo).*
- Any 2n-periodic orbit's ending point (x,y) will satisfy x+y=n
- All (x,y) coordinates marking the end of an orbit have the property that x ≡ y (mod 3)
  - For example: (1,1), (3,9), (1,4)

Counting Orbits (equilateral triangle)

- Fixing *n* this gives a bijection between the set of period 2*n* orbits and the set

  A_n = { ordered integer pairs (x,y) satisfying the conditions:
  - x+y=n
  - x≡y (mod 3)
  - y >= x >= 0}

e.g. for n=12, we have (0,12), (6,6), (3,9)

n	$\lvert A_n \rvert$
1	0
2	1
3	1
4	1
5	1
6	1
7	2
8	1

Combinatorics

- There exists a **bijection** between A_n and the **integer partitions** of n into 2's and 3's.

Integer partition into 2's and 3's: e.g. for n=12:

{ (3+3+3+3) , (3+3+2+2+2) , (2+2+2+2+2+2) }

correspond one-to-one to

{ (0, 12) , (6,6) , (3,9) }

Hard to count $\{$ orbits of period $2n$ $\}$

$\Updownarrow$

Easier to count

$A_n = \{ (x,y) : x+y=n,\ x \equiv y \pmod 3),\ y >= x >= 0 \}$

$\Updownarrow$

Familiar combinatoric problem! ☺

$\{$ integer partitions of n in 2's and 3's $\}$

Future Goals

- Create a useful coordinate system for shapes other than equilateral triangles.
- Classify the different orbits within these shapes.
- Find a strategy for counting the orbits.