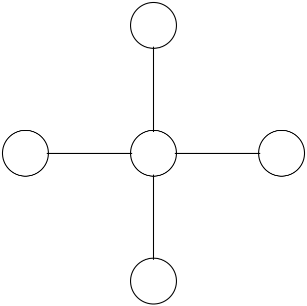
Graphs, Linear Algebra, and Finite Geometry

Dr. Gene Fiorini, Jaime Guerrero

July 14, 2011

Spoke Graphs

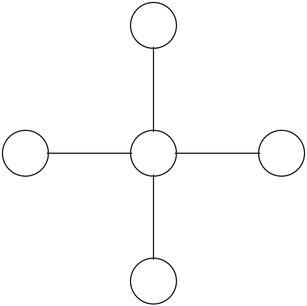
- Look like fireworks



Spoke Graphs

- Look like fireworks
- Adjacency matrices obey a certain form

$$A = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$



Spoke Adjacency Matrices

Theorem (Spokes and Even Powers)

Given a spoke graph $G = (V, E)$ with root degree $d \geq 3$, G's adjacency matrices are of the form

$$A^n = \begin{pmatrix} d^b & 0 & \cdots & 0 \\ 0 & d^{b-1} & \cdots & d^{b-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & d^{b-1} & \cdots & d^{b-1} \end{pmatrix}$$

where $n = 2b$, and $b \in \mathbb{Z}$.

Proof.

By induction on b. □

Spoke Adjacency Matrices

Theorem (Spokes and Odd Powers)

Given a spoke graph $G = (V, E)$ with root degree $d \geq 3$, G's adjacency matrices are of the form

$$A^n = \begin{pmatrix} 0 & d^b & \cdots & d^b \\ d^b & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ d^b & 0 & \cdots & 0 \end{pmatrix}$$

where $n = 2b + 1$, $d > 2$ and $b \in \mathbb{Z}$.

Proof.

By induction on b. $\square$

Spoke Eigenstuff

Theorem (Spoke Eigenvalues)

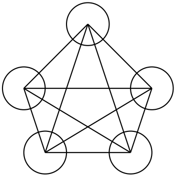
Given a star graph with root degree $d \geq 3$, the largest eigenvalue Λ of any of G's adjacency matrices is given by $\Lambda = \sqrt{d^n}$, where n is the exponent of the matrix in question.

Proof.

By induction. $\square$

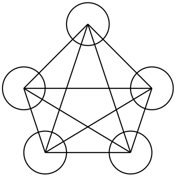
Complete Graphs and Matrices

- One of the simplest and richest structures available.



Complete Graphs and Matrices

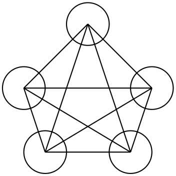
- One of the simplest and richest structures available.
- Each vertex is connected to every other vertex.



Complete Graphs and Matrices

- One of the simplest and richest structures available.
- Each vertex is connected to every other vertex.
- Pattern for powers is easy; general form is hard

$$A = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}$$



Complete Eigenstuff

Theorem (Characteristic Polynomial)

Given a complete graph K_n with degree d and adjacency matrix A raised to some power i

$$det(A^i - \lambda I) = (-1)^{d+1}(\lambda - d^{d+1})(\lambda + (-1)^{i+1})^d$$

Proof.

Hard. Sander notes that K_n has an eigenvalue of $n - 1 = d$. Use the properties of orthogonal matrices to construct each term individually. $\square$

Complete Eigenstuff

Corollary (Largest Eigenvalue)

Given a complete graph K_n with degree d and adjacency matrix A raised to some power i

$$\Lambda = d^i$$

Finite Geometry

- Geometry with a finite number of points and lines.

Finite Geometry

- Geometry with a finite number of points and lines.
- What interesting geometries can we find in graphs?
- What are the necessary and sufficient conditions for geometries to exist?

Finite Geometry

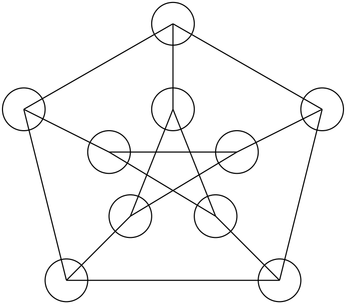
- Geometry with a finite number of points and lines.
- What interesting geometries can we find in graphs?
- What are the necessary and sufficient conditions for geometries to exist?
- Start with regular graphs.

Finite Geometry

- Geometry with a finite number of points and lines.
- What interesting geometries can we find in graphs?
- What are the necessary and sufficient conditions for geometries to exist?
- Start with regular graphs.
- Given K_n, let K_{n-1} be a line, and a graph's edge be a point.

The Petersen Graph

- The counterexample to everything.



The Petersen and Finite Geometry

- Line: Disjoint 5-cycles. Point: ?

The Petersen and Finite Geometry

- Line: Disjoint 5-cycles. Point: ?
- Cool stuff, but how do they intersect?

The Petersen and Finite Geometry

- Line: Disjoint 5-cycles. Point: ?
- Cool stuff, but how do they intersect?
- Gene: let lines be disjoint 5-cycles, and let points be perfect machings of the graph.

An Attempt

Theorem (Cycles and Perfect Matchings)

Given any perfect matching A for a Petersen Graph $\mathcal{P}$, $\mathcal{P} \setminus A$ forms two disjoint 5-cycles.

Theorem (Edges in Common)

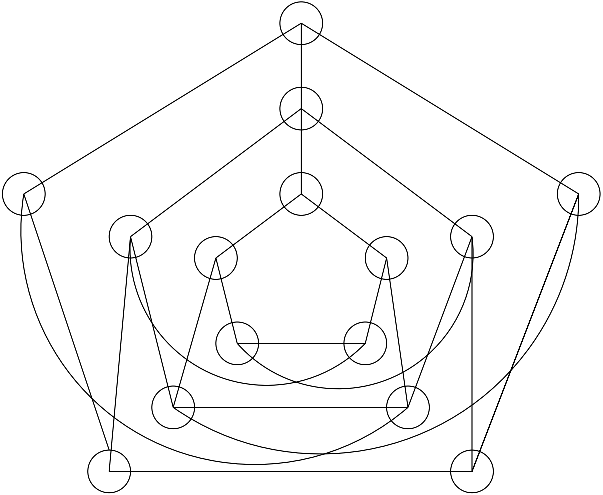
Given any two perfect matchings A and B for a Petersen Graph $\mathcal{P}$, A and B must share one edge in $\mathcal{P}$.

Theorem (Existence of Perfect Matchings)

There exist only two perfect matchings for $\mathcal{P}$.

Extending the Space

- Concentric Petersen Graphs!



A Theorem

Theorem (Perfect Matchings and Cycles)

Given concentric Petersen Graphs with r rings, remove $3k, k \in \mathbb{N}$ consecutive rings. Each vertex on the outside ring is the origin of a cycle of length $2r$.

Proof.

Construct the permutation groups, and count the number of "bridges". □

Final Thoughts

- Goal wasn't exactly reached.

Final Thoughts

- Goal wasn't exactly reached.
- Hope to continue this as a computer science project.