Hamiltonian Mechanics and Symplectic Geometry

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Background

In the previous talk, I demonstrated how the Lagrangian formulation greatly simplifies Newtonian mechanics. Today I will talk about Hamiltonian mechanics, specifically the geometry associated with this formulation.

I will first talk about this geometry restricted to vector spaces and linear transformations before discussing Hamiltonian systems in general (which are not linear).

Hamiltonian Mechanics

In Hamiltonian Mechanics, given a dynamical system, we describe it using n position and n momentum variables, $q_1, q_2, \ldots, q_n$ and $p_1, p_2, \ldots, p_n$. The phase space, P of a system is the topological space of all possible states of the system, that is, 2n-tuples $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. There is a function $H(\mathbf{q}, \mathbf{p}) : P \to \mathbb{R}$ called the Hamiltonian Using H, we can view the trajectories of the system as solutions to the equation:

$$\begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = -J_0 \begin{pmatrix} \frac{\partial H}{\partial \mathbf{q}} \\ \frac{\partial H}{\partial \mathbf{p}} \end{pmatrix}$$

where $J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$

Symplectic Structure of the Phase Space

Suppose we want to make a change of variables on the Phase space above.

Example: Let $\begin{pmatrix} \tilde{\mathbf{q}} \\ \tilde{\mathbf{p}} \end{pmatrix} = A \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}$

Be a linear change of variables on P. We would like Hamilton's equations to hold true in these new coordinates. But

$$\begin{pmatrix} \tilde{\mathbf{q}} \\ \tilde{\mathbf{p}} \end{pmatrix} = A \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = -J_0 A \begin{pmatrix} \frac{\partial H}{\partial \mathbf{q}} \\ \frac{\partial H}{\partial \mathbf{p}} \end{pmatrix} = -A J_0 A^T \begin{pmatrix} \frac{\partial H}{\partial \tilde{\mathbf{q}}} \\ \frac{\partial H}{\partial \tilde{\mathbf{p}}} \end{pmatrix}$$

With the last equality because the gradient transforms according to $\nabla H(\mathbf{q}, \mathbf{p}) = A^T \nabla H(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$.

Symplectic Matrices and Symplectic forms

Definition : A $2n \times 2n$ matrix A is symplectic if $AJ_0A^T = J_0$.

We can interpret the preceeding definition differently by considering any 2n-Dimensional vector space V and introducing a the bilinear form $\omega_0 : V \times V \to \mathbb{R}$ defined by $\omega_0(v, w) = \langle v, J_0 w\rangle$. This is known as the standard symplectic form.

From this point of view symplectic matrices are transforms that preserve ω_0, i.e. A is symplectic iff $\omega_0(Av, Aw) = \omega_0(v, w)$

Geometry of Symplectic Vector Spaces

What geometric properties do symplectic transforms exhibit?

- If A is a symplectic matrix, then A has determinant +1. Keeping in mind that the determinant of an n$\times$n matrix is the area of the parallelotope spanned by its columns, we can see that linear symplectic transforms preserve volume and orientation
- By looking closer at $\omega_0(v, w)$, we can see that it actually defines a geometric quantity assocaited with the parallelogram P spanned by v and w, called the *symplectic area* of P. Since symplectic matrices preserve the action of ω_0, they preserve symplectic area.

Connection with the Hamiltonian Phase Space

In general, Hamiltonian phase spaces are **not** symplectic vector spaces, but symplectic manifolds. There is a strong analogy between these objects, however. Symplectic manifolds are smooth, even-dimensional manifolds equipped with a special 2-form called a symplectic form (the differential form equivalent of ω_0)

The notion of the preservation of volume by symplectic transforms gives rise to a deeper result about the time evolution of solutions in the phase space.

Theorem: (Liouville's theorem) The natural volume form on a symplectic manifold is constant along Hamiltonian flows.

Future Directions

Although I have mostly studied symplectic vector spaces, I plan on continuing to study differential geometry and symplectic manifolds. I have also been looking into minimal surfaces and their connection with energy-minimization (and hence Hamiltonians) and have the longterm goal of eventually finding some interface between these two fields of study.

Short-term goals now include using Liouville's theorem to understand Noether's theorem on conserved currents and using symplectic geometry to better understand the Euler-Lagrange equations.