PERFECTLY MATCHED LAYERS AND THE ART OF CLOAKING

SECOND PRESENTATION

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Outline

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III. Wave Equation in 1st order form with PML

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Wave Equation

- The wave equation is the following partial differential equation (PDE) :

$$u_{tt} = c^2\, \nabla \cdot (\nabla u)$$

- In Cartesian coordinates:

$$u_{tt} = c^2\, u_{xx}, \qquad \text{for } u(t,x)$$
$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right), \qquad \text{for } u(t,x,y)$$

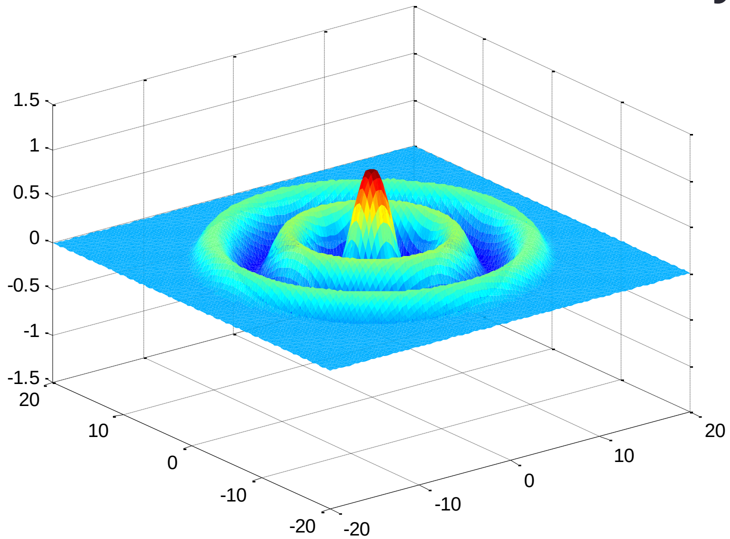
- In polar coordinates:

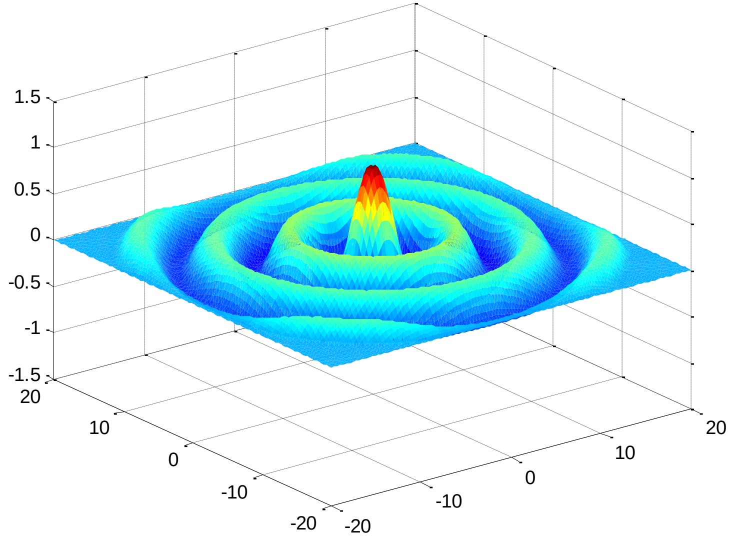
$$u_{tt} = c^2\left(\frac{1}{r}\,(r\,u_r)_r + \frac{1}{r^2}\,u_{\theta\theta}\right)$$

- These equations produce the familiar traveling wave solutions.

Perfectly Matched Layers

- PML is a method for adding an absorbing layer, often adjacent to the boundary, which applies an exponential decay to waves traveling in the region.
- Hence, PML can be used to numerically simulate the wave equation locally on an “infinite domain” (truly the equation is on a finite domain, but there are minimal affects from the boundary.)





Perfectly Matched Layers

- The technique involves a coordinate stretching given by

$$x_i \to \tilde{x}_i \equiv x_i + \frac{1}{s} \int_0^{x_i} \sigma_i(x)\, dx$$

where s comes from the Laplace transform

$$\hat{u} = \int_0^{\infty} e^{-st}\, u\, dt$$

- To apply this to our PDEs, we translate this coordinate stretching to partial derivatives

$$\frac{\partial}{\partial \tilde{x}_i} = \frac{1}{1 + \frac{\sigma_i(x_i)}{s}} \frac{\partial}{\partial x_i}$$

Wave Equation in 1st order form with PML

- In the 1D case, the wave equation separates to
$$u_{tt} = c^2\, u_{xx} \quad \Longleftrightarrow \quad u_t = c\, v_x, v_t = c\, u_x$$
- If we consider the coordinate stretching on the first of the two coupled PDEs, after Laplace transforming we have
$$s\,\hat{u} = \frac{c}{1 + \frac{\sigma(x)}{s}} \hat{v}_x$$
$$s\,\hat{u} + \sigma(x)\,\hat{u} = c\,\hat{v}_x$$
$$u_t = c\, v_x - \sigma(x)\, u$$
- Hence, our equations are
$$u_t = c\, v_x - \sigma(x)\, u$$
$$v_t = c\, u_x - \sigma(x)\, v$$

Wave Equation in 2nd order form with PML

- If we try to apply the same technique to the 2D wave equation, we get frustrating "cross terms".
- In Grote and Sim's *Efficient PML*, they show the algebra necessary to effectively keep the wave equation in its second order form. This results in:

$$\text{1D}: \quad u_{tt} + \sigma(x)\, u_t = c^2\, u_{xx} + \phi_x$$
$$\phi_t = -\sigma(x)\phi - c^2\sigma(x)\, u_x$$

$$\text{2D}: \quad u_{tt} + (\sigma_1 + \sigma_2)\, u_t + \sigma_1\sigma_2 u = c^2\,(u_{xx} + u_{yy}) + \boldsymbol{\phi}_x$$
$$\boldsymbol{\phi}_t = -\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \boldsymbol{\phi} - c^2 \begin{pmatrix} \sigma_1 - \sigma_2 & 0 \\ 0 & \sigma_2 - \sigma_1 \end{pmatrix} \boldsymbol{\nabla} u$$

Polar Wave Equation with PML

- Again the wave equation in polar coordinates:

$$u_{tt} = c^2 \left(\frac{1}{r} (r\, u_r)_r + \frac{1}{r^2}\, u_{\theta\theta} \right)$$

- If we perform the stretching in only the r coordinate, we are able to show that PML for the polar wave equation is

$$u_{tt} + \frac{2}{r}\, \alpha(r)\, u_t + \left(\frac{\alpha(r)}{r} \right)^2 u = \left(\frac{c}{r} \right)^2 (\Psi + u_{\theta\theta})$$

$$r\, u_{rt} + \alpha(r)\, u_r = \phi_t + \sigma(r)\, \phi$$

$$r\, \phi_{rt} + \alpha(r) \phi_r = \Psi_t + \sigma(r)\, \Psi$$

where $\alpha(r) = \int_{[0,r]} \sigma(s)\, ds$

- We needed two auxiliary functions because of the r terms in the original PDE.

Uniting the One Dimensional Cases

1st order form with PML

$$u_t = c\, v_x - \sigma(x)\, u$$
$$v_t = c\, u_x - \sigma(x)\, v$$

??

2nd order form with PML

$$u_{tt} + \sigma(x)\, u_t = c^2\, u_{xx} + \phi_x$$
$$\phi_t = -\sigma(x)\phi \; - c^2\sigma(x)\, u_x$$

Uniting the One Dimensional Cases

1st order form with PML

$$u_t = c\, v_x - \sigma(x)\, u$$
$$v_t = c\, u_x - \sigma(x)\, v$$

$$\phi = -c\, \sigma(x)\, v$$

2nd order form with PML

$$u_{tt} + \sigma(x)\, u_t = c^2\, u_{xx} + \phi_x$$
$$\phi_t = -\sigma(x)\phi \, - c^2 \sigma(x)\, u_x$$

Uniting the One Dimensional Cases

1st order form with PML

$$u_t = c\, v_x - \sigma(x)\, u$$
$$v_t = c\, u_x - \sigma(x)\, v$$

??

2nd order form with linear dampening

$$u_{tt} + \sigma(x)\, u_t = c^2\, u_{xx}$$

$$\phi = -c\, \sigma(x)\, v$$

??

2nd order form with PML

$$u_{tt} + \sigma(x)\, u_t = c^2\, u_{xx} + \phi_x$$
$$\phi_t = -\sigma(x)\phi \;- c^2\sigma(x)\, u_x$$

Uniting the One Dimensional Cases

1st order form with PML

$$u_t = c\, v_x - \sigma(x)\, u$$
$$v_t = c\, u_x - \sigma(x)\, v$$

Drop a term

1st order form with linear dampening

$$u_t = c\, v_x - \sigma(x)\, u$$
$$v_t = c\, u_x$$

$$\phi = -c\, \sigma(x)\, v$$

2nd order form with PML

$$u_{tt} + \sigma(x)\, u_t = c^2\, u_{xx} + \phi_x$$
$$\phi_t = -\sigma(x)\phi \; - c^2 \sigma(x)\, u_x$$

2nd order form with linear dampening

$$u_{tt} + \sigma(x)\, u_t = c^2\, u_{xx}$$

Soliton Solution to One Dimensional Case

- A Soliton is a traveling wave solution to a wave-like PDE. Typical examples include the *Korteweg-deVries* equation which describes water waves in a trough or canal.
- If we take our 1D 1st order wave equation with PML,

$$u_t = c\,v_x - \sigma(x)\,u$$
$$v_t = c\,u_x - \sigma(x) v, \qquad x \in (-\infty, \infty), t \in (0, \infty)$$
$$u(t = 0, x) = f(x), \qquad u_t(t = 0, x) = 0$$

- And assume the form $u = F(x + c\,t)\,g(x)$. Then after some algebra we find our solution is

$$u(t, x) = \frac{f(x + c\,t)}{b(x + c\,t) + \frac{1}{b(x + c\,t)}}\,b(x) + \frac{f(x - c\,t)}{b(x - c\,t) + \frac{1}{b(x - c\,t)}}\frac{1}{b(x)}$$

where $b(x) = \exp\left(\int_{[0,x]} \frac{\sigma(s)}{c}\,ds\right)$.

Stability of Finite Difference Methods

- I chose to use second order FDMs to discretize PDEs.
- Suppose we are considering the problem

$$u_{tt} + \sigma(x)\, u_t = c^2\, u_{xx} + \phi_x$$
$$\phi_t = -\sigma(x)\phi \; - c^2\sigma(x)\, u_x$$

- We define our mesh by $\Delta x, \Delta t$ such that

$$U_j^i \approx u(i\,\Delta t, j\,\Delta x) \;\text{ and }\; \phi_j^i \approx \phi(i\,\Delta t, j\,\Delta x)$$

for $i = 0, \dots, n-1,\ j = 0, \dots, m-1.$

- The standard discretization of the second equation however, is unstable (ϕ explodes)

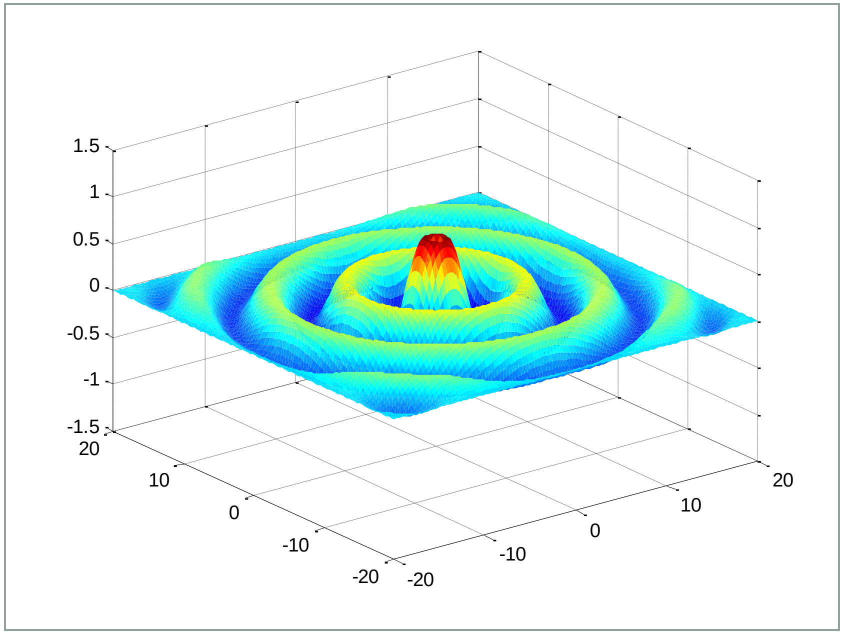
$$\frac{\phi_j^{i+1} - \phi_j^{i-1}}{2\,dt} = -\sigma_j\, \phi_j^i - c^2\, \sigma_j \frac{u_{j+1}^i - u_{j-1}^i}{2\,dx}$$

Stability of Finite Difference Methods

- We can experimentally observe that the instability in begins in ϕ before affecting u. Hence for stability concerns we can consider just
$$\phi_t = -\sigma(x_0)\,\phi$$
- Which is very easy to analyze with Von Neumann Stability analysis, to show that
$$\frac{\phi^{i+1} - \phi^{i-1}}{2\,dt} = -\sigma\,\phi^i$$
accrues error with each step on the order of
$$g = dt\,\sigma + \sqrt{(dt\,\sigma)^2 + 1}$$
- However, if we use
$$\frac{\phi^{i+1} - \phi^{i-1}}{2\,dt} = -\sigma\,\frac{\phi^i + \phi^{i+1}}{2}$$
then our error remains fairly constant ($g = 1$).
- Each PDE has a term which needs this substitution for stability.

Results and Future Work

- I have over 7000 lines of MATLAB code detailing computationally efficient FDMs to each of the following PDEs
  - 1D Wave Equation
    - 1st order form
    - 2nd order form
    - 1st order form with linear dampening
    - 1st order form with PML
    - 2nd order form with PML
  - 2D Wave Equation
    - 1st order form
    - 1st order form with PML in one direction
    - 2nd order form
    - 2nd order form with PML
  - Polar Wave Equation
    - 2nd order form
    - 2nd order form with PML
  - 1D Helmholtz Equation
  - 2D Helmholtz Equation



- I haven't seen polar wave equation with PML or the soliton solution for the 1D case online, though they aren't that difficult to determine.
- For future work, I'd like to understand PML with the Helmholtz equation as well as determine analytic properties of the solution to the wave equation with PML.

References

I. J.-P. Bérenger, “A perfectly matched layer for the absorption of electromagnetic waves,” J. Comput. Phys., vol. 114, no. 1, pp. 185–200, 1994.

II. M. Grote & I. Sim, “Efficient PML for the wave equation,” Jan. 2010. [arXiv:1001.0319] [math.NA]

III. S. Johnson, “Notes on Perfectly Matched Layers (PML),” Aug. 2007. math.mit.edu/~stevenj/18.369/pml.pdf

IV. M. Lassas & E. Somersalo, “On the existence and convergence of the solution of PML equations,” J. Computing, vol. 60, Issue 3, pp. 229-241, 1998.