Unique Decomposability of Graph Gluings

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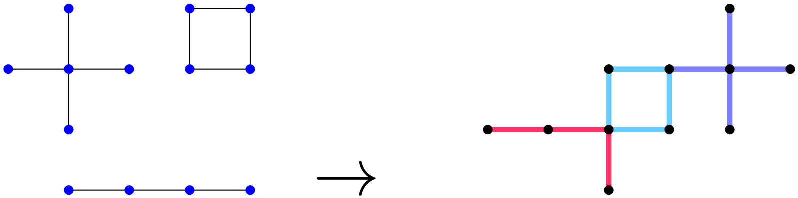
Mentor: Dr. Robert DeMarco

June 6, 2013

Statement of Project

Can the components of a multi-component graph be 'glued' together in such a way that the resultant graph is 'uniquely decomposable' into a collection of graphs similar to the original components?



Gluing of Graphs

Definition (B. DeMarco and A. Redlich)

A tuple $(G_1, G_2, \ldots, G_k, H, H_1, H_2, \ldots, H_k)$ is a *gluing* of $G_1, \ldots, G_k$ if:

- H is a connected graph,
- $H_1, \ldots, H_k$ are subgraphs of H,
- $H_i \sim G_i$ for all $i \in \{1, 2, \ldots, k\}$, and
- $\cup_{i=1}^{k} E(H_i) = E(H)$.

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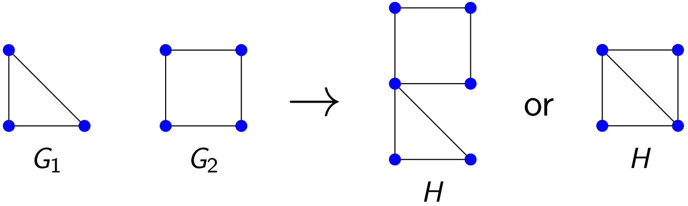
Example:



Unique Decomposition

Definition (B. DeMarco and A. Redlich)

A tuple $(G_1, \ldots, G_k, H)$ is *uniquely decomposable* if there exists only one tuple $H_1, \ldots, H_k$ such that $(G_1, \ldots, G_k, H, H_1, \ldots, H_k)$ is a gluing.

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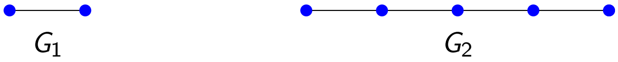

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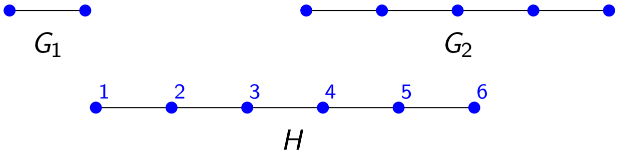

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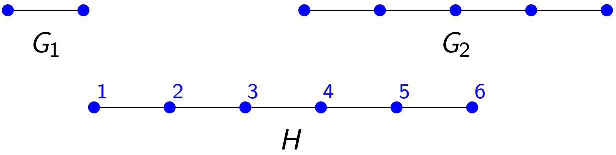

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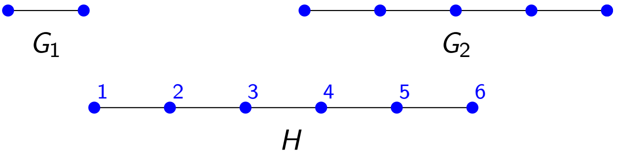

(G_1, G_2, H) not uniquely decomposable:

$H_1 = \{1, 2\}$ and $H_2 = \{2, 3, 4, 5, 6\}$

or

$H_1 = \{5, 6\}$ and $H_2 = \{1, 2, 3, 4, 5\}$

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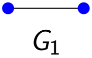

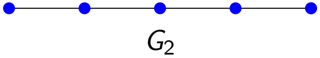

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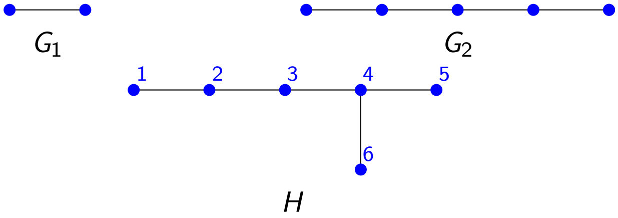

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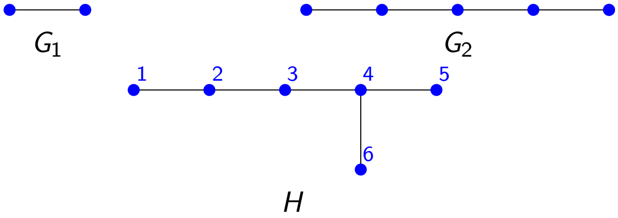

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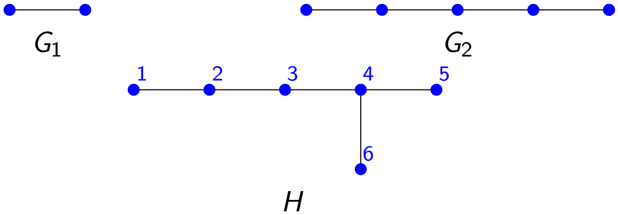

(G_1, G_2, H) not uniquely decomposable:

$H_1 = \{4, 5\}$ and $H_2 = \{1, 2, 3, 4, 6\}$

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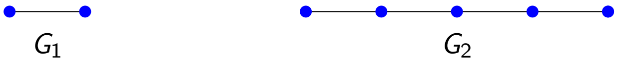

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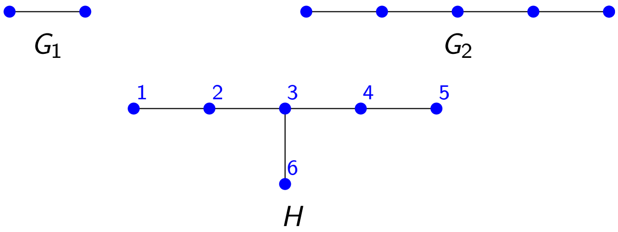

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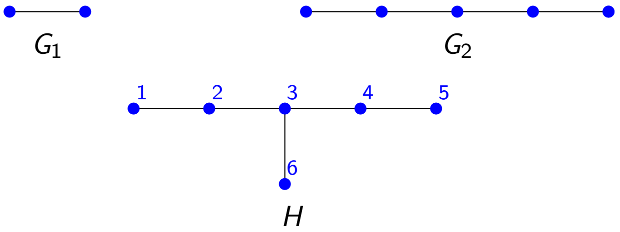

(G_1, G_2, H) is uniquely decomposable:

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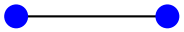
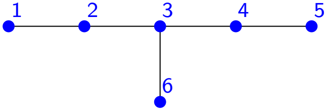


G_1

G_2



H

(G_1, G_2, H) is uniquely decomposable:
$H_1 = \{3, 6\}$ and $H_2 = \{1, 2, 3, 4, 5\}$

Gluing of Two-Component Graphs

Theorem (DeMarco and Redlich)

If $G_1 \neq G_2$, neither G_1 nor G_2 is a single vertex, $\{G_1, G_2\} \neq \{P_1, P_2\}$ and $\{G_1, G_2\} \neq \{P_1, P_3\}$, there exists a graph H such that (G_1, G_2, H, H_1, H_2) is a uniquely decomposable gluing and $H \neq G_1, G_2$. Furthermore, H may be constructed explicitly.

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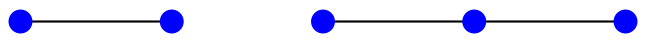
So, the only (interesting) two-component graphs that do not admit a unique composition are:



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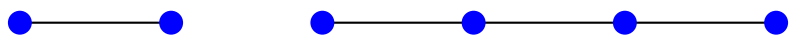

Goals for the Summer

- Investigate the gluing of multi-component graphs
  - 3-component graphs
  - Graphs whose components are trees
- Understand the context of this problem
  - Random graph theory

Reference

B. DeMarco and A. Redlich, Graph decomposition and parity (2012); http://arxiv.org/abs/1211.2243.