

DIMATIA/MCW Report

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1 Talks

The first week that we were here, there were talks given by some professors from Charles University. Two of those talks were especially interesting to me. The first was one given by Jiří Fiala on interval graphs. An interval graph is the graph which represents the intersection of intervals along a real line and a given graph is an interval graph if and only if it has an interval model. Fiala explained to us that the model of an interval graph can be simplified because the usual model of intersecting intervals contains unneeded information. For instance, an interval model involving n intervals can be represented as a sequence of at most $n - 1$ cliques in which each vertex appears in a consecutive subsequence of cliques. Fiala also informed us of the various practical applications of interval graphs, including scheduling problems wherein the maximum independent set represents the maximum number of tasks which can be done without conflict and the colouring of a graph minimizes the number of machines needed to complete a task.

What really intrigued me about Fiala's talk was when he discussed the problem of finding Hamiltonian paths, cycles, and k -staves on interval graphs. While not every interval graph has a Hamiltonian path, $K_{1,3}$ for instance is an interval graph with no Hamilton path, Fiala explained that if an interval graph has a Hamiltonian path, then it must also have a monotone Hamiltonian path beginning at vertex v_1 and ending at v_n . This claim is a consequence of a lemma by Peter Damaschke which gives a means of constructing a monotone path from two vertex disjoint paths. Fiala then went through the algorithm that Damaschke presents in [1] to find a spanning k -stave in an interval graph. After going through the proof of the algorithm and discussing the relationship between scattering number and the existence of spanning k -staves, Fiala left us with two problems to think about. The first was the problem of finding a Hamiltonian path in an interval graph with one fixed endpoint and the second was the problem of finding a Hamiltonian path in an interval graph with two fixed endpoints. These problems were very interesting to me and although I was not able to make much progress toward solving them, I learned a lot while trying.

The other topic which interested me was the one presented to us by Andrew Goodall. He gave two talks on counting the proper k -colourings of graphs and acyclic orientations of graphs. I was familiar with the definition of a proper k -colouring of a graph, G , as a function, $f : V(G) \rightarrow [k]$, such that if $uv \in E(G)$, $f(u) \neq f(v)$. However, I had never considered

counting the number of such colourings and was quite intrigued when Goodall introduced the following definition:

Definition 1. *The chromatic polynomial of a graph, G , counts the number of proper k -colourings of the vertices of G and is denoted by $P(G; k)$.*

Therefore, $P(G; k) > 0$ if and only if G has a proper k -colouring. Goodall then went on to explain that a proper colouring of G defines a unique acyclic orientation of G whereby $u \rightarrow v$ if $f(u) < f(v)$ because f is a total order. Therefore, the longest directed path that can exist in this orientation is a P_k . This motivated us to prove the following theorem about the converse of this statement:

Theorem 1. *If G has an acyclic orientation in which all dipaths have length less than k , $P(G; k) > 0$.*

To prove this statement, we needed that every acyclic orientation has at least one source and one sink. This follows from the fact that if P_r with $r \leq k$ is the longest dipath in G , then the first vertex in this path can only have edges leaving from it and the last vertex in this path can only have edges entering. The graph G can now be shown to have a proper $r \leq k$ -colouring as follows. Colour all sinks in G with the colour r (since no edges can exist between sinks) and then delete these vertices. The resultant subgraph still has an acyclic orientation (if it did not, this would imply G did not have an acyclic orientation) with maximum dipath P_{r-1} and so we can colour all sinks in this subgraph with $r-1$. Continuing on in this fashion, it is clear that G can be given a proper r -colouring and, since $r \leq k$, it follows that $P(G; k) > 0$.

Goodall then talked about the chromatic polynomial in more depth. We were able to deduce that $P(\bar{K}_n; k) = k^n$ and that $P(K_n; k) = k^{\underline{n}}$, where $k^{\underline{n}}$ denotes the falling factorial $k(k-1) \cdots (k-n+1)$ by greedily colouring the vertices. After some explanation, we arrived at the following definition of the chromatic polynomial:

Definition 2. $P(G; k) = \sum_{r=1}^k a_r(G) k^{\underline{r}}$, where $a_r(G)$ is the number of ways to partition the vertices of G into r non-empty stable sets.

This definition makes sense because for each partition of the vertices into r non-empty stable sets, there are $k^{\underline{r}}$ ways to assign colours, since vertices in each set can receive the same colour.

We then established the following recurrence relation for the chromatic polynomial of a graph, G with some non-edge, $xy \notin E(G)$:

$$P(G; k) = P(G + xy; k) + P(G \cdot xy; k).$$

This recurrence is true because the number of colourings of G is the number of colourings of G where $f(x) \neq f(y)$ plus the number of colourings of G where $f(x) = f(y)$. The first is the number of colourings of the graph G' where $xy \in E(G)$ and the second is the number of colourings of the graph G'' where x and y are identified as a single vertex. From here,

we were able to arrive at the following recurrence by setting $H = G + xy$, $G = H \setminus xy$, and $G \cdot xy = H/xy$, where $H \setminus xy$ denotes the deletion of the edge xy and H/xy denotes the contraction of the edge xy :

$$P(H; k) = P(H \setminus xy; k) - P(H/xy; k).$$

The first recurrence is useful for dense graphs and the second is useful for sparse graphs.

Goodall then left us with several exercises to attempt, several of which I will present in the following section.

A few days later, Goodall returned to talk to us more about acyclic orientations and the chromatic polynomial. We began by reviewing the chromatic polynomials of cycles and trees on n vertices (proofs in next section). They are as follows :

$$P(C_n; k) = (k - 1)^n + (-1)^n(k - 1)$$

$$P(T_n; k) = k(k - 1)^{n-1}.$$

We then discussed the coefficients of terms in the chromatic polynomial. We will denote the coefficient of the term k^r by $[k^r]P(G; k)$. Goodall explained that the deletion-contraction recurrence could be used to show that for all graphs G on n vertices without loops, $[k^n]P(G; k) = 1$. To find $[k^{n-1}]P(G; k)$, we returned to our definition of the chromatic polynomial as $P(G; k) = \sum_{i=1}^n a_i(G)k^i$. We saw that $a_n(G) = 1$, since there is only one way to partition $V(G)$ into n stable sets (each vertex goes into its own set). We also saw that $a_{n-1}(G) = \binom{n}{2} - |E(G)|$ because the only way partition $V(G)$ into $n - 1$ sets is for one set to contain two vertices and the rest to be singletons. There are $\binom{n}{2}$ choices for the set which contains two vertices, however $|E(G)|$ of these will be an edge and therefore will not be stable. From here, we were able to show that

$$[k^{n-1}]P(G; k) = -\left(\sum_{i=1}^{n-1} i\right)a_n(G) + a_{n-1}(G) = -\binom{n}{2} + \binom{n}{2} - |E(G)| = -|E(G)|.$$

We then began to calculate the coefficient of k^{n-2} , but ran into problems when we could not count the number of partitions of vertices into $n - 2$ stable sets. I was very interested in this, however, and so I looked at [2] in which Meredith proves that if a graph G with n vertices and m edges has a girth of $n - s + 1$ and p cycles of this length, then for $r > s$, $|[k^r]P(G; k)| = \binom{m}{n-r}$ and $|[k^s]P(G; k)| = \binom{m}{n-s} - p$. This implies that for any graph on at least three vertices, $[k^{n-2}]P(G; k) = \binom{m}{2} - p$, where p is the number of triangles in G . Additionally, for any triangle-free graph on at least four vertices, $[k^{n-3}]P(G; k) = \binom{m}{3} - q$, where q is the number of C_4 's in G . Beyond these small values, however, it seems quite difficult to compute the coefficients of terms in the chromatic polynomial.

After discussing these coefficients, we went on to define the following polynomial:

$$Q(G; k) = (-1)^n P(G; -k) = \sum_{i=1}^n b_i(G)k^{n-i},$$

where $b_i(G)$ counts the number of subgraphs of G with i edges containing no broken cycles. Goodall pointed out that if the girth of $G = g$, then $b_i(G) = \binom{m}{i}$ for $0 < i < g - 1$ and $b_{g-1}(G) = \binom{m}{g-1} - q$, where q is the number of g -cycles. This makes sense because if the smallest cycle in G is a C_g , then any subgraph on up to $g - 2$ vertices cannot possibly contain a broken cycle, and so the number of subgraphs of G with $i \leq g - 2$ edges without a broken cycle is just the number of subgraphs of G with i edges. Additionally, for each subgraph on $g - 1$ vertices, there is only one way to have a broken cycle for each cycle of length g . These two formulas correspond exactly with Meredith's theorem in [2]. The point of this discussion was to bring us back to the topic of acyclic orientation of graphs. Goodall explained that the evaluation of $Q(G; 1) = (-1)^n P(G; -1) = \sum b_i(G)$ calculates the number of acyclic orientations of G . He then asked us to think about a combinatorial interpretation for why the number of acyclic orientations of a graph should be the same as the total number of subgraphs with no broken cycles. I thought about this problem for quite some time and this question was the topic of my final presentation. I present one thought below.

In [3] when Stanley proves that $(-1)^n P(G; -1)$ is the number of acyclic orientations of G , he also shows that the number of acyclic orientations of G are the number of equivalence classes for a particular equivalence relation defined as follows:

Definition 3 (Stanley 1973). *Let G be a graph on n vertices and let σ be a bijective mapping $\sigma : V(G) \rightarrow [n]$. Then if $\sim$ is an equivalence relation on the set of all σ labelings of G with the condition that $\sigma \sim \sigma'$ if whenever $uv \in E(G)$, then $\sigma(u) < \sigma(v) \Leftrightarrow \sigma'(u) < \sigma'(v)$, $(-1)^n P(G; -1)$ is the number of equivalence classes of this relation.*

This made me wonder whether there may be some sort of relation that could be defined on G for which the subgraphs with no broken cycles determine the equivalence classes for the relation. I have not yet been successful in finding one, however, and I have not yet been able to come up with a combinatorial interpretation for why the number of acyclic orientations of a graph is the same as the number of subgraphs with no broken cycles.

Finally, Goodall went through a proof for why the function $Q(G; 1) = A(G)$, where $A(G)$ is the number of acyclic orientations of G . This involved using Q to show that the deletion-contraction recurrence held for A . Goodall left us with many problems to work out, several of which I present solutions to in the next section.

2 Exercises

1. Find the chromatic polynomial for a tree on n vertices, T_n .

The chromatic polynomial for a tree on n vertices is

$$P(T_n; k) = k(k - 1)^{n-1}.$$

Proof. When $n = 2$, it is clear that there are k colours available for the first vertex and $k - 1$ colours available for the second vertex, giving a chromatic polynomial of $k(k - 1)$. So, assume that up to some m , the chromatic polynomial for a given tree on m vertices is

$k(k-1)^{m-1}$. Now consider a tree on $m+1$ vertices. By the deletion-contraction recurrence, we have that

$$\begin{aligned}
P(T_{m+1}; k) &= P(T_{m+1} \setminus e \in E(T); k) - P(T_{m+1}/e \in E(T); k) \\
&= P(T_m + K_1; k) - P(T_m; k) \\
&= k(P(T_m; k)) - P(T_m; k) \\
&= P(T_m; k)(k-1) \\
&= k(k-1)^m.
\end{aligned}$$

□

2. Find the chromatic polynomial for the cycle on n vertices, C_n .

The chromatic polynomial for a cycle on n vertices is

$$P(C_n; k) = (k-1)^n + (-1)^n(k-1).$$

Proof. When $n = 3$, it is clear that there are k colours available for the first vertex, $k-1$ colours available for the second vertex, and $k-2$ colours available for the third vertex. Therefore, $P(C_3; k) = k(k-1)(k-2) = k^3 - 3k^2 + 2k = (k-1)^3 + (-1)^3(k-1)$. So, assume that up to some m , the chromatic polynomial for the cycle on m vertices is $(k-1)^m + (-1)^m(k-1)$. Now, consider the cycle on $m+1$ vertices. By the deletion-contraction recurrence, we have that

$$\begin{aligned}
P(C_{m+1}; k) &= P(T_{m+1}; k) - P(C_m; k) \\
&= k(k-1)^m - [(k-1)^m + (-1)^m(k-1)] \\
&= k(k-1)^m - (k-1)^m + (-1)^{m+1}(k-1) \\
&= (k-1)^m(k-1) + (-1)^{m+1}(k-1) \\
&= (k-1)^{m+1} + (-1)^{m+1}(k-1).
\end{aligned}$$

□

3. Find the chromatic polynomial for the wheel graph on $n+1$ vertices, W_n .

To find the chromatic polynomial for the wheel graph, we will first start with the graph H which consists of the wheel graph W_n with an extra vertex attached to some 3-clique in W_n . Note that once the vertices of W_n are coloured, there are $k-3$ colours left for the additional vertex. Therefore, $P(H; k) = (k-3)P(W_n; k)$. We now apply the deletion-contraction recurrence to H where the edge used is the edge, e , adjacent to the two degree four vertices in H . Note that $P(H \setminus e; k) = P(W_{n+1}; k)$ and $P(H/e; k) = (k-2)P(W_{n-1}; k)$, since H/e is the graph which consists of W_{n-1} with an additional vertex adjacent to a degree three vertex and the degree $n-1$ vertex. This is sufficient to establish the following recursive definition:

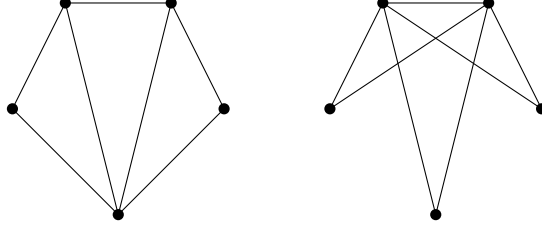
$$P(W_n; k) = (k-3)P(W_{n-1}; k) + (k-2)P(W_{n-2}; k).$$

This has the characteristic polynomial $r^2 - (k-3)r - (k-2) = 0$ with roots $k-2$ and -1 . Therefore, the chromatic polynomial for W_n must be of the form $P(W_n; k) = A(k-$

$2)^n + B(-1)^n$. Using as initial conditions $P(W_3; k) = k(k-1)(k-2)(k-3)$ and $P(W_4; k) = k^5 - 8k^4 + 24k^3 - 31k^2 + 14k$ (calculated using the deletion-contraction relation described above), we get that

$$P(W_n; k) = k(k-2)^n + k(k-2)(-1)^n.$$

4. Find two non-isomorphic graphs G and G' such that $P(G; k) = P(G'; k)$.



Each of these graphs has a chromatic polynomial of $k(k-1)(k-2)^3$. This can be seen by using a greedy colouring. In the first graph, there are k choices for how to colour the degree 4 vertex. Then, there are $k-1$ choices for the next vertex, and $k-2$ colours for each subsequent vertex because vertices can be selected so that they are adjacent to two adjacent and already coloured vertices. In the second graph, there are k choices for how to colour one degree 4 vertex and then $k-1$ choices for the second degree 4 vertex. Since none of the remaining vertices are adjacent to one another but are all adjacent to the degree 4 vertices, there are $k-2$ choices for how to color each of them.

This particular example can be extended into two infinite families whereby for each $n \geq 5$, the graphs on n vertices are not isomorphic but have the same chromatic polynomial. One family consists of a cycle on n vertices which is triangulated in such a way that one vertex is of degree $n-1$. The other family is a connected graph on n vertices which has $n-2$ triangles such that each triangle shares one common edge, creating two degree $n-1$ vertices. Both families have chromatic polynomials of $k(k-1)(k-2)^{n-2}$ as can be easily seen by extending the greedy colourings described above.

5. By considering the definition of the chromatic polynomial, prove that

$$k^n = \sum_{1 \leq i \leq n} S(n, i) k^i,$$

where $S(n, i)$ is equal to the number of partitions of an n -set into i non-empty sets.

Proof. We know that $P(\bar{K}_n; k) = k^n = \sum_{1 \leq i \leq n} a_i(G) k^i$, where $a_i(G)$ is the number of partitions of the vertices of G into i non-empty stable sets. Since any set of vertices of $\bar{K}_n$ forms a stable set, $a_i(\bar{K}_n) = S(n, i)$. Therefore, $k^n = \sum_{1 \leq i \leq n} S(n, i) k^i$. $\square$

6. If G is a connected graph, what does the coefficient of k in the chromatic polynomial $P(G; k)$ count?

Expanding the terms of $P(G; k) = \sum_{i=1}^n a_i(G) k^i$, it can be seen that $[k]P(G; k) = \sum_{i=1}^n (-1)^{i+1} (i-1)! a_i(G)$, where $a_i(G)$ denotes the number of partitions of the vertex set of

G into i non-empty stable sets. However, this gives no information about the meaning of the coefficient of the chromatic polynomial. So, we seek other interpretations.

Since $(k - 1)$ is a factor of $P(G; k)$ whenever G has an edge (this can be shown by the deletion-contraction relation), $P(G; 1) = 0$. Therefore, the sum of the coefficients of the chromatic polynomial must equal 0, and so $[k]P(G; k) = -\sum_{i=2}^n [k^i]P(G; k)$. Again, this definition is unsatisfying as it sheds no light on how to interpret this coefficient.

However, using the definition of the chromatic polynomial presented by Whitney in [4] of $P(G; k) = \sum_i (-1)^i b_i k^{n-i}$, we can clearly see that when we take $i = n - 1$, $[k]P(G; k) = (-1)^{n-1} b_{n-1}$, where b_{n-1} is the number of subgraphs of G with $n - 1$ edges which contain no broken cycles.

Interestingly enough, there is another equivalent definition of the chromatic polynomial which gives an alternative definition of the coefficient of k . Goodall and Nešetřil leave as a problem to show that for a connected graph G , $P(G; k) = \sum_{i=0}^{n-1} (-1)^i c_i(G) k^{n-i}$, where $c_i(G)$ is the number of cocliques of order $n - i$ occurring as leaf nodes in the computation tree for G . This interpretation of the chromatic polynomial implies that $[k]P(G; k) = (-1)^{n-1} c_{n-1}(G)$. In absolute value, this is the number of cocliques of order 1 which occur as leaf nodes in the computation tree for G .

These two different interpretations of the coefficient of k in the chromatic polynomial make me wonder whether there is some connection between the number of cocliques of order 1 which occur as leaf nodes in the computation tree for G and the number of subgraphs of G on $n - 1$ edges which contain no broken cycles. It seems that there should be some combinatorial interpretation for why these two values are equivalent. Additionally, it must also be true that the number of acyclic orientations of G is the number of cocliques of any order occurring as leaf nodes in the computation tree of G . Perhaps this may give another combinatorial explanation for the number of acyclic orientations of G .

References

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