Unique Decomposability of Gluings of Three Component Tree-Like Graphs

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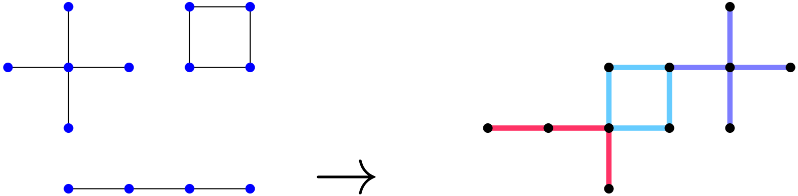
Statement of Project

Can the components of a multi-component graph be ‘glued’ together in such a way that the resultant graph can be ‘uniquely decomposed’ into a collection of graphs isomorphic to the original graph?



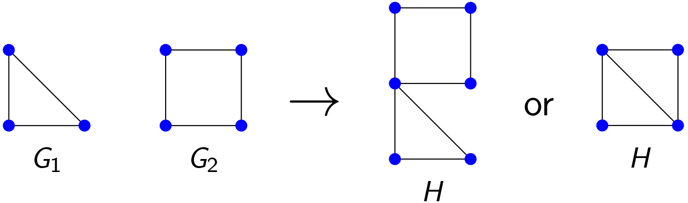
Gluing of Graphs

Definition (B. DeMarco and A. Redlich)

A tuple $(G_1, G_2, \ldots, G_k, H, H_1, H_2, \ldots, H_k)$ is a *gluing* of $G_1, \ldots, G_k$ if:

- H is a connected graph,
- $H_1, \ldots, H_k$ are subgraphs of H,
- $H_i \sim G_i$ for all $i \in \{1, 2, \ldots, k\}$, and
- $\cup_{i=1}^{k} E(H_i) = E(H)$.

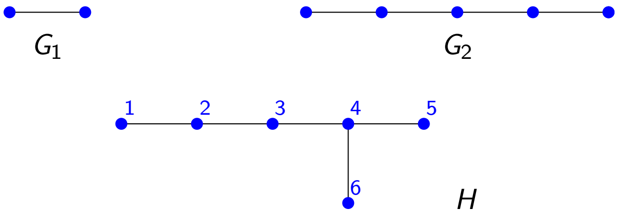
Example:



Unique Decomposition

Definition (B. DeMarco and A. Redlich)

A tuple $(G_1, \ldots, G_k, H)$ is *uniquely decomposable* if there exists only one tuple $H_1, \ldots, H_k$ such that $(G_1, \ldots, G_k, H, H_1, \ldots, H_k)$ is a gluing.

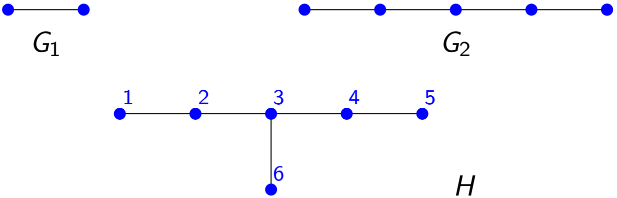


(G_1, G_2, H) not uniquely decomposable:
$H_1 = \{4, 5\}$ and $H_2 = \{1, 2, 3, 4, 6\}$ or $H_1 = \{4, 6\}$ and $H_2 = \{1, 2, 3, 4, 5\}$

Unique Decomposition

Definition (B. DeMarco and A. Redlich)

A tuple $(G_1, \ldots, G_k, H)$ is *uniquely decomposable* if there exists only one tuple $H_1, \ldots, H_k$ such that $(G_1, \ldots, G_k, H, H_1, \ldots, H_k)$ is a gluing.

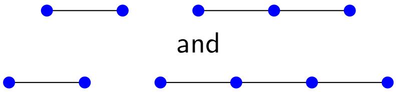


(G_1, G_2, H) is uniquely decomposable:

$$H_1 = \{3, 6\} \text{ and } H_2 = \{1, 2, 3, 4, 5\}$$

Gluings of Two-Component Graphs

- The case of two-component graphs has been done (B. DeMarco and A. Redlich)
  - The only (interesting) two-component graphs that do not admit a unique decomposition are:



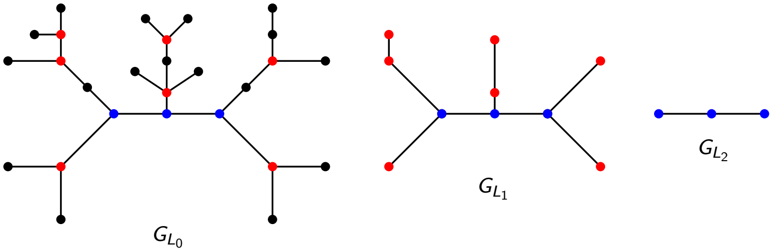
- Proof for two-component graphs used idea of "block degree"
- Tried to use block degree to extend to three-component graphs
- Problems arose in some cases
- Decided to look at "depth degree"

Depth Degree of Trees

Definition

The graph of a tree G in level i, denoted G_{L_i}, is defined as follows:

- $G_{L_0} = G$
- $v \in V(G_{L_i})$ if $\deg_{G_{L_{i-1}}}(v) \geq 3$
- $uv \in E(G_{L_i})$ if for any $w \in V(G_{L_{i-1}})$ for which there is a $u - w - v$ path in $G_{L_{i-1}}$, $\deg_{G_{L_{i-1}}}(w) < 3$.



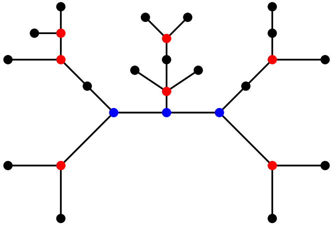
Depth Degree of Trees

Definition

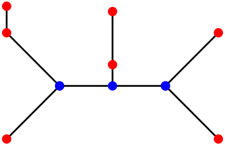
The depth degree of a vertex v in a tree G, denoted $t_G(v)$, is the maximum level to which v survives.

Definition

The depth degree of a tree G, denoted $T(G)$, is the maximum level to which any vertex in G survives.



G_{L_0}



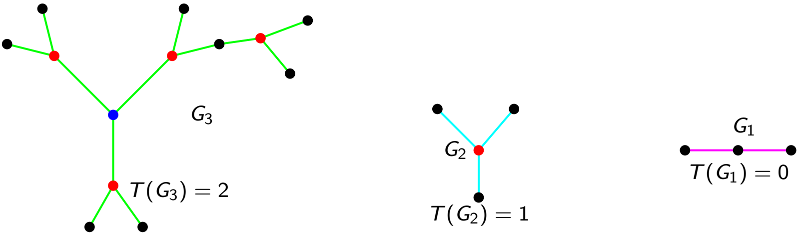
G_{L_1}

G_{L_2}

Gluings of Three-Component Trees

- Found unique gluings for all three-component trees except four cases where unique gluings do not exist:
  - P_2, P_3, P_4
  - P_2, P_3, P_5
  - P_2, P_4, P_5
  - P_2, P_4, P_6
- Gluings characterized in 15 cases based on tree depth of components, number of vertices of maximum tree depth, and length of path, for those components which are paths.

Gluings of Three-Component Trees: $T(G_3) > T(G_2) > T(G_1)$



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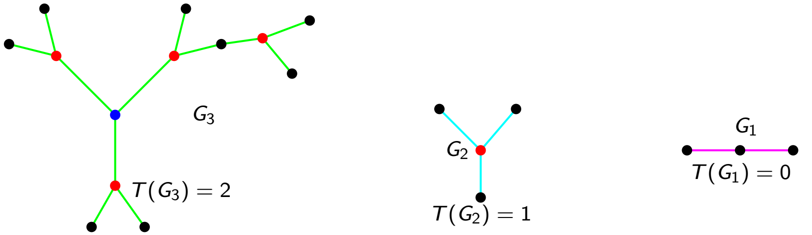

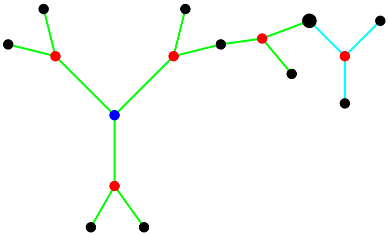

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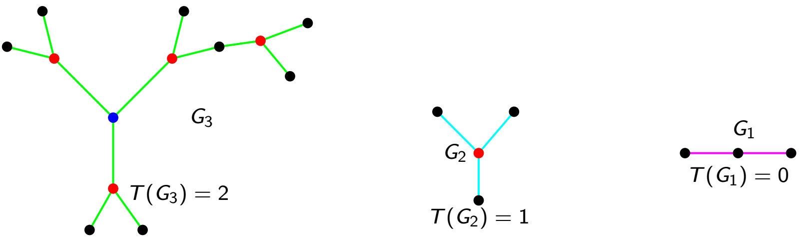

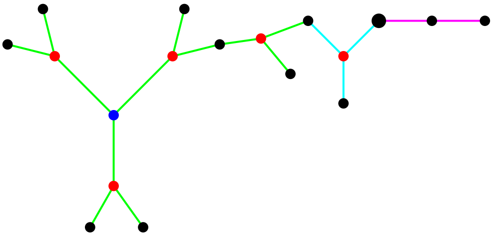

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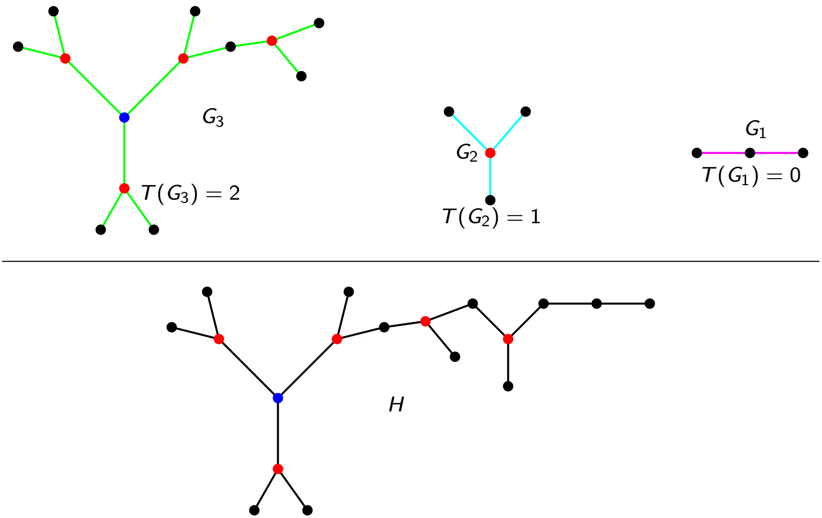

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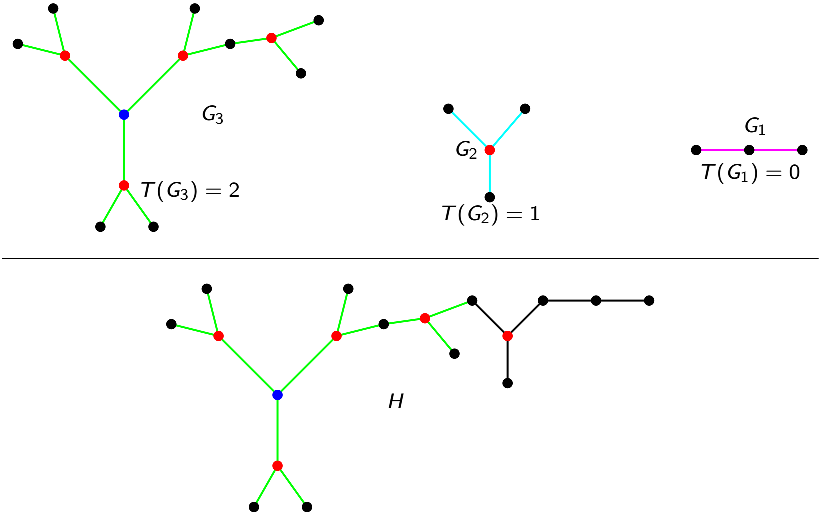

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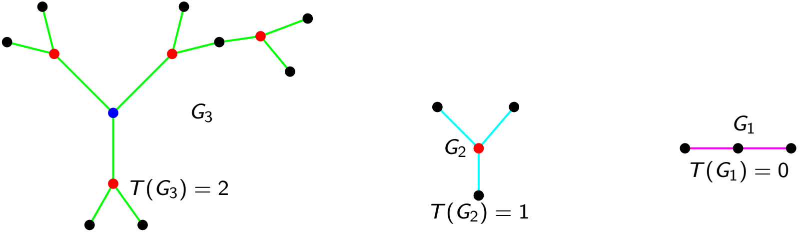

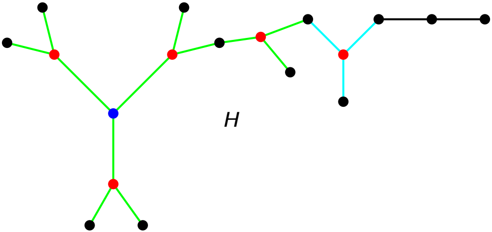

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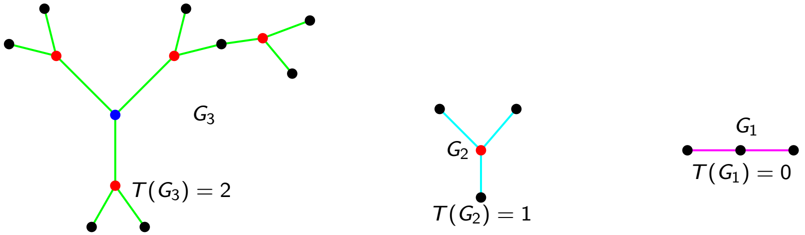
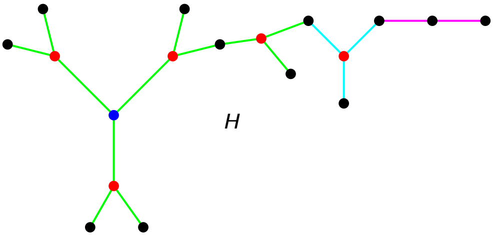
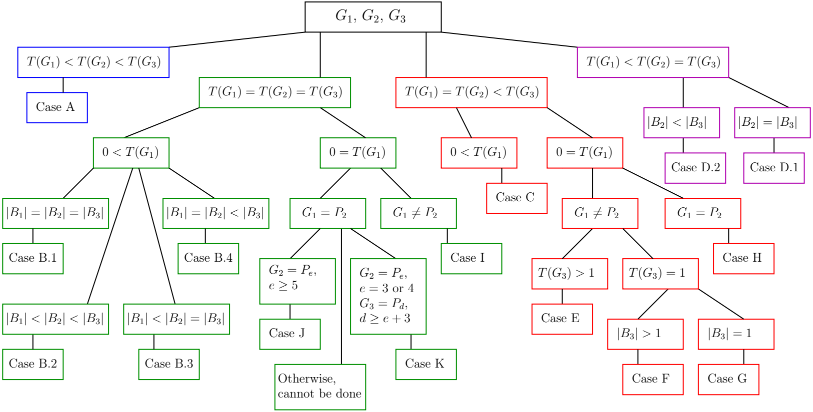




Gluing of Three-Component Trees



Future Work

- Gluings of three-component graphs where the components are not trees
- Gluings of graphs with any number of components which are trees