GENERALIZED METRIC SPACES ARE RAMSEY

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Abstract. In this note we give Ramsey expansions of classes of generalized metric spaces. This complements result of Conant about the extension properties for partial isomorphisms and extends earlier results authors result giving Ramsey property of convexly ordere S-metric spaces.

Dedicated to old friend Norbert Sauer.

1. Introduction

Given $S \subseteq \mathbb{R}_{>0}$ (that is, the set of positive reals) an *S-metric space* is an metric space with all distances contained in S. The following 4-values condition characterises when the class of all finite S-metric spaces is closed for amalagamation:

Definition 1.1 (Delhommé, Laflamme, Pouzet, Sauer [2])**.** *A subset $S \subseteq \mathbb{R}_{>0}$ satisfies* 4-values condition*, if for every $a, b, c, d \in S$ if there is some $x \in S$ such that triangles with distances a–b–x and c–d–x satisfies the triangle inequality there is also $y \in S$ such that that triangles with distances a–c–y and b–d–y satisfies the triangle inequality.*

or equivalently by the associativity of the following operation: for $a, b \in S$ denote by $a \oplus_S b = \sup\{x \in S; x \leq a + b\}$.

Theorem 1.2 (Sauer [8])**.** *A subset S of the positive reals satisfies the 4-values condition if and only if the operation $\oplus_S$ is associative.*

Sauer [7] used the equivalence above to show those S such that there exists an countable (ultra)homogeneous S metric space.

The above notions naturally lead to generalized metric spaces where distance sets forms a monoid as defined by Conant [1] and introduced bellow. In this note we give Ramsey expansions of generalized metric spaces. The following is a main results of this note complements result of Conant [1] about the extension properties for partial isomorphisms and extends earlier results about Ramsey property of S-metric spaces [4].

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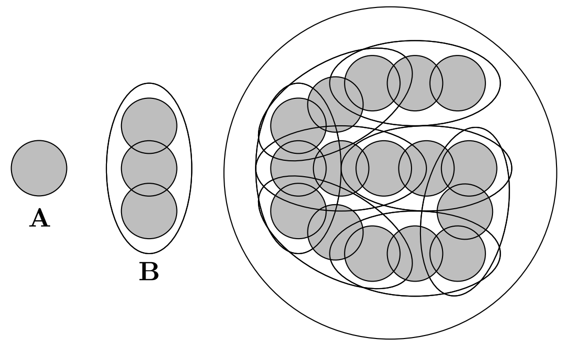

FIGURE 1. Construction of a Ramsey object by multiamalgamation.

Theorem 1.3. *If $\mathfrak{M} = (M, \oplus, \preceq, 0)$ is a semi-archimedean distance monoid where $\preceq$ is linear order, then $\overrightarrow{\mathcal{M}}_{\mathfrak{M}}$ of convexly ordered finite $\mathfrak{M}$-metric spaces has the Ramsey property.*

Definition 1.4 (Conant [1])**.**

Warning: Connant has linear order

(1) *A structure $\mathfrak{M} = (M, \oplus, \preceq, 0)$ is a* distance monoid *if*
   (a) *$(M, \preceq, 0)$ is a lattice with least element 0;*
   (b) *$(M, \oplus, 0)$ is a commutative monoid with identity 0;*
   (c) *for all $r, s, t, u \in M$, if $r \preceq s$ and $t \preceq u$ then $r \oplus t \preceq s \oplus u$.*
(2) *Suppose $\mathfrak{M} = (M, \oplus, \preceq, 0)$ is a distance monoid. Given a set A and a function $d : A \times A \to M$, we call (A, d) an* **M**-metric space *if*
   (a) *for all $a, b \in A$, $d(a, b) = 0$ if and only if $a = b$;*
   (b) *for all $a, b \in A$, $d(a, b) = d(b, a)$;*
   (c) *for all $a, b, c \in A$, $d(a, c) \preceq d(a, b) \oplus d(b, c)$.*
(3) *Given a distance monoid $\mathfrak{M}$, we let $\mathcal{M}_{\mathbf{M}}$ denote the class of finite* **M***-metric spaces.*

Given $n \geq 0$ and $r \in M$ we denote by $n \times r$ a sequence $r \oplus r \oplus \cdots \oplus r$ of length n.

Definition 1.5. *A distance monoid $\mathfrak{M} = (M, \oplus, \preceq, 0)$ is* archimedean *if, for all $r, s \in M^{>0}$, there exists some integer $n > 0$ such that $s \preceq n \times r$.*

Definition 1.6. *A distance monoid $\mathfrak{M} = (M, \oplus, \preceq, 0)$ is* semi-archimedean *if, for all $r, s \in M^{>0}$, if $n \times r < s$ for all $n > 0$ then $r \oplus s = s$.*

2. Previous work — multiamalgamation

We now refine amalgamation classes. Our aim is to describe even stronger sufficient criteria for Ramsey classes.

First, we develop a generalised notion of amalgamation which will serve as a useful tool for the construction of Ramsey objects. As schematically depicted in Figure 1, Ramsey objects are a result of amalgamation of multiple copies of a given structure which are all performed at once. In a non-trivial class this leads to many problems. Instead of working with complicated amalgamation diagrams we split the amalgamation into two steps—the construction of (up to isomorphism unique) free amalgamation (which yields an incomplete or "partial" structure) followed then by a completion. Formally this is done as follows:

Definition 2.1. *An L-structure $\mathbf{A}$ is* irreducible *if $\mathbf{A}$ is not a free amalgam of two proper substructures of $\mathbf{A}$.*

Remark: In the case of relational structures this definition allows a combinatorial description: $\mathbf{A}$ is irreducible if for every pair of distinct vertices u, v there is a tuple $\vec{t} \in R_{\mathbf{A}}$ (of some relation $R \in L_{\mathcal{R}}$) such that $\vec{t}$ contains both u and v. In the case of structures with function symbols this is more complicated as we have to consider closures defined below.

Thus the irreducibility is meant with respect to the free amalgamation. The irreducible structures are our building blocks. Moreover in structural Ramsey theory we are fortunate that most structures are (or may be interpreted as) irreducible. And in the most interesting case, the structures may be completed to irreducible structures. This will be introduced now by means of the following variant of the homomorphism notion.

Definition 2.3. *A homomorphism $f : \mathbf{A} \to \mathbf{B}$ is* homomorphism-embedding *if f restricted to any irreducible substructure of $\mathbf{A}$ is an embedding to $\mathbf{B}$.*

While for (undirected) graphs the homomorphism and homomorphism-embedding coincide, for structures they differ. For example any homomorphism-embedding of the Fano plane into a hypergraph is actually an embedding.

Definition 2.4. *Let $\mathbf{C}$ be a structure. An irreducible structure $\mathbf{C}'$ is a* completion *of $\mathbf{C}$ if there exists a homomorphism-embedding $\mathbf{C} \to \mathbf{C}'$.*

Let $\mathbf{B}$ be an irreducible substructure of $\mathbf{C}$. We say that irreducible structure $\mathbf{C}'$ is a completion of $\mathbf{C}$ with respect to copies of $\mathbf{B}$ *if there exists a function $f : C \to C'$ such that for every $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ the function f restricted to $\widetilde{B}$ is an embedding of $\widetilde{\mathbf{B}}$ to $\mathbf{C}'$.*

If $\mathbf{C}'$ belongs to a given class $\mathcal{K}$, then $\mathbf{C}'$ is called a $\mathcal{K}$-completion of $\mathbf{C}$ with respect to copies of $\mathbf{B}$.

Remark: Completion may be seen as a generalised form of amalgamation. To see that let $\mathcal{K}$ be a class of irreducible structures. The amalgamation property of $\mathcal{K}$ can be equivalently formulated as follows: For $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and α_1 embedding of $\mathbf{A}$ into $\mathbf{B}_1$, α_2 embedding of $\mathbf{A}$ into $\mathbf{B}_2$, there is $\mathbf{C} \in \mathcal{K}$ which is a completion of the free amalgamation (which itself is not necessarily in $\mathcal{K}$) of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2.

Free amalgamation may result in a reducible structure. The pairs of vertices where one vertex belong to $\mathbf{B}_1 \setminus \alpha_1(\mathbf{A})$ and the other to $\mathbf{B}_2 \setminus \alpha_2(\mathbf{A})$ are never both contained in a single tuple of any relation. Such pairs can be thought of as *holes* and a completion is then a process of filling in the holes to obtain irreducible structures (in a given class $\mathcal{K}$) while preserving all embeddings of irreducible structures.

Completion with respect to copies of $\mathbf{B}$ is the weakest notion of completion which preserve the Ramsey property for given structures $\mathbf{A}$ and $\mathbf{B}$. Note that in this case f does not need to be a homomorphism-embedding (and even homomorphism).

We now state all necessary conditions for our main result stated in this section:

Definition 2.6. *Let L be a language, $\mathcal{R}$ be a Ramsey class of finite irreducible L-structures. We say that a subclass $\mathcal{K}$ of $\mathcal{R}$ is an* $\mathcal{R}$-multiamalgamation class *if the following conditions are satisfied:*

(1) Hereditary property: *For every $\mathbf{A} \in \mathcal{K}$ and a substructure $\mathbf{B}$ of $\mathbf{A}$ we have $\mathbf{B} \in \mathcal{K}$.*
(2) Strong amalgamation property: *For $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and α_1 embedding of $\mathbf{A}$ into $\mathbf{B}_1$, α_2 embedding of $\mathbf{A}$ into $\mathbf{B}_2$, there is $\mathbf{C} \in \mathcal{K}$ which contains a strong amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2 as a substructure.*
(3) Locally finite completion property: *Let $\mathbf{B} \in \mathcal{K}$ and $\mathbf{C}_0 \in \mathcal{R}$. Then there exists $n = n(\mathbf{B}, \mathbf{C}_0)$ such that if closed L-structure $\mathbf{C}$ satisfies the following:*
    (a) *there is a homomorphism-embedding from $\mathbf{C}$ to $\mathbf{C}_0$, (in other words, $\mathbf{C}_0$ is a completion of $\mathbf{C}$), and,*
    (b) *every substructure of $\mathbf{C}$ with at most n vertices has a $\mathcal{K}$-completion.*
    Then there exists $\mathbf{C}' \in \mathcal{K}$ that is a completion of $\mathbf{C}$ with respect to copies of $\mathbf{B}$.

We can now state the main result of [4] as:

Theorem 2.7 ([4])**.** *Every $\mathcal{R}$-multiamalgamation class $\mathcal{K}$ is Ramsey.*

The proof of this result is not easy and involves interplay of several key constructions of structural Ramsey theory particularly Partite Lemma and Partite Construction (see [4] for details).

3. Proof of the main theorem

We first show some basic facts about completion into $\mathfrak{M}$-metric spaces and then show the desired Ramsey property. We follow presentation of [4] which is in many way similar to analysis in [1].

Given distance monoid $\mathfrak{M} = (M, \oplus, \preceq, 0)$ we interpret an $\mathfrak{M}$-metric space as a relational structure $\mathbf{A}$ in the language $L_{\mathfrak{M}}$ with (possibly infinitely many) binary relations R^s, $s \in M^{>0}$, where we put, for every $u \neq v \in A$, $(u, v) \in R^l_{\mathbf{A}}$ if and only if $d(u, v) = l$. We do not explicitly represent that $d(u, u) = 0$ (i.e. no loops are added).

Definition 3.1. *Every $L_{\mathfrak{M}}$-structure where all relations are symmetric and irreflexive and every pair of vertices is in at most one relation is* $\mathfrak{M}$-graph *which we define as a graph with edges labelled by M^0. Every non-induced substructure of an $\mathfrak{M}$-metric space is* $\mathfrak{M}$-metric graph *($\mathfrak{M}$-metric graphs are structures with have a strong completion to $\mathfrak{M}$-metric space in the sense of Definition 2.4). Every $\mathfrak{M}$-graph that is not $\mathfrak{M}$-metric graph is* non-$\mathfrak{M}$-metric graph.

The universal ultrahomogeneous metric space was constructed by Urysohn [9] (by a Fraïssé type construction thus anticipating Fraïssé by more than 20 years). Generalising this result, when $\mathcal{M}_{\mathfrak{M}}$ forms an amalgamation class, we call this Fraïssé limit an *Urysohn $\mathfrak{M}$-metric space.*

The following corollary summarise the results of [7] and [8] which are important for our construction. For the completeness we include a proof.

Proposition 3.2 ([1]). *Let $\mathfrak{M}$ be a distance monoid.*

(1) *Let $\mathbf{G}$ be a finite $\mathfrak{M}$-metric graph. Denote by $d'(u,v)$ the minimal $\mathfrak{M}$-length of a walk from u to v (that is the $\oplus$ sum of all distances along the walk) and by $\overrightarrow{W}(u,v)$ the corresponding walk. Then $\mathbf{G}$ can be completed to an $\mathfrak{M}$-metric space $\mathbf{A}$ by putting for every pair $u \neq v \in G$, $d_{\mathbf{A}}(u,v) = d'(u,v)$.*

(2) *An $\mathfrak{M}$-graph $\mathbf{G}$ is a $\mathfrak{M}$-metric graph if and only if all of its cycles are $\mathfrak{M}$-metric.*

Proof. Both statements can be seen as an easy consequences of the associativity of $\oplus$:
(1). First assume that $\mathbf{G}$ is $\mathfrak{M}$-metric. We show that the completion described will give $\mathfrak{M}$-metric space. We verify that d' satisfies the triangle inequality. Assume, to the contrary, the existence of vertices u, v, w such that $d'(u,v) > d'(u,w) + d'(w,v)$. Combine the walks $\overrightarrow{W}(u,w)$ and $\overrightarrow{W}(w,v)$ to the walk $\vec{p}$. By the associativity of $\oplus$ the $\mathfrak{M}$-length of $\vec{p}$ is $d'(u,w) \oplus d'(w,v) \succeq d'(u,v)$. It follows that $d'(u,v) \preceq d'(u,w) \oplus d'(w,v)$ which is a contradiction.

We have shown that d' forms an $\mathfrak{M}$-metric space on vertices of $\mathbf{G}$. We still need to check that $d_{\mathbf{G}}(u,v) = d'(u,v)$ whenever $d_{\mathbf{G}}(u,v)$ is defined. We show a stronger claim: let $\mathbf{B}$ be a completion of $\mathbf{G}$ to an $\mathfrak{M}$-metric space then $d_{\mathbf{B}}(u,v) \preceq d'(u,v)$ for every $u \neq v \in G$.

We proceed by the induction on n which is the length of definition $\mathfrak{M}$-walk. For $n = 3$ this follows from the triangle inequality. For $n > 3$ denote by $(p_1, p_2, \ldots, p_{n-1}, p_n)$ the walk $\overrightarrow{W}(u,v)$. By the induction hypothesis we know that $d_{\mathbf{B}}(u, p_{n-1}) \preceq d'(u, p_{n-1})$. The inequality then follows from the associativity of $\oplus$ and the triangle inequality. This finishes the proof of statement (1).
(2). Assume that $\mathbf{G}$ is non-$\mathfrak{M}$-metric. In this case we have a pair of vertices u and v with the distance defined such that $d'(u,v) < d_{\mathbf{A}}(u,v)$. Because $\oplus$ is monotonous, it is easy to see that the walk $\overrightarrow{W}(u,v)$ can be turned into a path. A non-metric cycle is induced on vertices of this path. This is a contradiction. $\square$

3.1. **Archimedian monoids.** In this section we strengthen results of [4] showing that for archimedian monoid $\mathfrak{M}$ the class $\overrightarrow{\mathcal{M}}_\mathfrak{M}$ is Ramsey.

Lemma 3.3. *Let $\mathfrak{M}$ be an finite archimedian distance monoid. Then there every non-$\mathfrak{M}$-metric cycle has at most $|\mathfrak{M}|^2$ vertices.*

Proof. Assume, to the contrary, that there is a non-$\mathfrak{M}$-metric cycle $\mathbf{C}$ with $n > |\mathfrak{M}|$ vertices. By Proposition 3.2 we know that $\mathbf{C}$ contains a pair of vertices whose distance is longer than the $\mathfrak{M}$-length of a path connecting them. We can thus order vertices of $\mathfrak{M}$ as $v_1, v_2, \ldots, v_n$, such that $d_\mathbf{C}(v_1, v_n) \succ v_1 \oplus v_2 \oplus \ldots \oplus v_n$. Because $n > |\mathfrak{M}|^2$ we know that there ss at least one distance ℓ repeating in the sequence $|\mathfrak{M}|$ times. On one hand we have that $|\mathfrak{M}| \times \ell \preceq v_1 \oplus v_2 \oplus \ldots \oplus v_n \prec d_\mathbf{C}(v_1, v_n)$. Because $\mathfrak{M}$ is archimedian we also have $m \times v \succeq d_\mathbf{C}(v_1, v_n)$ for some m and it is easy to see that this m can be chosen so $m \leq |\mathfrak{M}|$ □

Corollary 3.4. *Let $\mathfrak{M}$ be an finite archimedian distance monoid. Then the class of all finite $\mathfrak{M}$-metric spaces with free, i.e. arbitrary, ordering of vertices, $\overrightarrow{\mathcal{M}}_\mathfrak{M}$, is a Ramsey class.*

Proof. By Proposition 3.2 the weakly ordered structure has a strong completion in $\overrightarrow{\mathcal{M}}_\mathfrak{M}$ if and only if all its cycles are $\mathfrak{M}$-metric. By Lemma 3.3 the set of non-$\mathfrak{M}$-metric cycles is finite and and the statement follow by Theorem 2.7. □

Corollary 3.5. *Let $\mathfrak{M}$ be archimedian distance monoid. Then the class of all finite $\mathfrak{M}$-metric spaces with free ordering of vertices, $\overrightarrow{\mathcal{M}}_\mathfrak{M}$, is a Ramsey class.*

Proof. Fix $\mathbf{A}, \mathbf{B} \in \overrightarrow{\mathcal{M}}_\mathfrak{M}$. Denote by M_0 the set of distances used in $\mathbf{B}$ (which is finite, because $\mathbf{B}$ is finite). It is easy to see that there is an finite archimedian submonoid $\mathfrak{M}'$ of $\mathfrak{M}$ containing M_0.

Write how?

Having finite archimedian distance monoid $\mathfrak{M}'$ and, both $\mathbf{A}$ and $\mathbf{B}$ are $\mathfrak{M}'$-metric spaces we apply Corollary 3.5. □

3.2. **Non-archimedian monoids.** The following is motivated by terminology of Sauer [7].

Definition 3.6. *Given distance monoid $\mathfrak{M}$, a* block *of $\mathfrak{M}$ is any maximal sub-monoid $\mathfrak{B}$ which is archimedian.*

Definition 3.7. *Given distance monoid $\mathfrak{M}$, $j \in \mathfrak{M}$ is* jump number *if $j \neq$ $\max(\mathfrak{M})$ and $j = \max(\mathfrak{B})$ for some block b.*

Lemma 3.8. *Let $\mathfrak{B}$ be block of $\mathfrak{M}$. Then*

1. $\max(\mathfrak{B}) \oplus \max(\mathfrak{B}) = \max(\mathfrak{B})$ *and*
2. *elements of $\mathfrak{B}^{>0}$ forms a linear interval of $\preceq$.*

Proof. (1) follows from maximality of $\mathfrak{B}$. To see (2) consider $0 \prec a \prec b \prec \max(\mathfrak{B})$ such that $a \in \mathfrak{B}$ and $b \notin \mathfrak{B}$. Chose maixmal such b. From maximality of $\mathfrak{B}$ it follows that $b \oplus b \leq \max(\mathfrak{B})$ and thus $b \oplus b = b$. Because $\oplus$ is monotone it follows that for every $c \prec b$ it hols that $c \oplus c \leq b$ which is a contradiction with $a \in \mathfrak{B}$. $\square$

It follows that jump numbers can, equivalently, be characterised as those numbers j such that $j \oplus j = j$.

Definition 3.9. *Let $\mathbf{A}$ be an $\mathfrak{M}$-metric space and $\mathfrak{B}$ block of $\mathfrak{M}$. We define a* block equivalence $\sim_{\mathfrak{B}}$ *on vertices of $\mathbf{A}$ by putting $u \sim_{\mathfrak{B}} v$ whenever $d(u, v) \leq \max(\mathfrak{B})$.*

It is easy to see that $\max(\mathfrak{B}) \oplus \max(\mathfrak{B}) = \max(\mathfrak{B})$ and for every $a, b \preceq \max(\mathfrak{B})$ it holds that $a \oplus b \preceq \mathfrak{B}$ and thus paths where every pair of vertices has distance at most $\max(\mathfrak{B})$ can only be completed by distances at most $\max(\mathfrak{B})$. Consequently $\sim_{\mathfrak{B}}$ is indeed an equivalence relation.

The following definition and technical lemma is the key to obtaining a locally finite description of $\mathcal{M}_{\mathfrak{M}}$ (needed for Theorem 2.7):

Definition 3.10. *Let $\mathbf{P}$ be a $\mathfrak{M}$-metric path. Denote by $B(\mathbf{P})$ the block of $\mathfrak{M}$ containing the maximal distance of an edge in $\mathbf{P}$. Let $\mathbf{P}'$ be any $\mathfrak{M}$-metric path. We say that $\mathbf{P}' \preceq_{\mathfrak{M}} \mathbf{P}$ if all the distances in $\mathbf{P}'$ are at most $\max(B(\mathbf{P}))$.*

Lemma 3.11. *Let $\mathfrak{M}$ be a finite distance monoid and let $\mathbf{C}$ be a non-$\mathfrak{M}$-metric cycle. Then there exists paths $\mathbf{P}^i$, $1 \leq i \leq p$, of vertices of $\mathbf{C}$ such that the cycle created by identifying all vertices of each path into a single vertex is non-$\mathfrak{M}$-metric cycle with at most $2|\mathfrak{M}|$ vertices, and moreover every cycle created from $\mathbf{C}$ by replacing each of the paths $\mathbf{P}^i$ by arbitrary path $\mathbf{P}'^i$, $\mathbf{P}'^i \preceq_{\mathfrak{M}} \mathbf{P}^i$ is non-$\mathfrak{M}$-metric.*

We will call paths in the family $\mathbf{P}^i$ *unimportant.*

Proof. Let $\mathbf{C}$ be a non-$\mathfrak{M}$-metric cycle with $n > |\mathfrak{M}|$ vertices. By Proposition 3.2 we know that $\mathbf{C}$ contains a pair of vertices whose distance is longer than the $\mathfrak{M}$-length of path connecting them. We can thus order vertices of $\mathfrak{M}$ as $v_1, v_2, \ldots, v_n$, such that $d_{\mathbf{C}}(v_1, v_n) > v_1 \oplus v_2 \oplus \ldots \oplus v_n$. Denote by

$$l_j = v_1 \oplus v_2 \oplus \ldots \oplus v_j$$

the $\mathfrak{M}$-length of the walk formed by the initial segment on j vertices. We know that $l_j \leq l_{j+1}$ for every $1 < j \leq n$.

A path induced by $\mathbf{C}$ on vertices $(v_j, v_{j+1}, \ldots v_k)$ is unimportant if $l_j = l_k$. The paths $\mathbf{P}^i$, $1 \leq i \leq p$, will consist of all inclusion maximal unimportant paths. As a special case, if is only one inclusion maximal unimportant path on vertices $v_3, v_4, \ldots v_n$, put $\mathbf{P}^1$ to be the path on vertices $v_3, v_4, \ldots, v_{n-1}$ so the result of identification is a triangle.

Because there are only finitely many values in $\mathfrak{M}$, we know that there are are at most $|\mathfrak{M}|$ choices of j such that $l_j < l_{j+1}$. We thus know that there are at most $|\mathfrak{M}|$ pairs v_j, v_{j+1} which are not part of an unimportant path. It follows that the graph created by concatenating all unimportant paths has at most $2|\mathfrak{M}|$ vertices (and, because of the special case, at least three vertices). It is also easy to verify that the resulting graph is non-$\mathfrak{M}$-metric cycle because the $\mathfrak{M}$-length of walk connecting v_1 and v_n was not affected by such replacements: if path $\mathbf{P}^i$ is unimportant then every path $\mathbf{P}'^i$, $\mathbf{P}'^i \preceq \mathbf{P}^i$ is also unimportant. $\square$

The following equivalences can be defined on $\mathfrak{M}$-metric spaces for non-archimedian $\mathfrak{M}$.

Definition 3.12 ([6])**.** *Given distance monoid $\mathfrak{M}$, denote by $J_{\mathfrak{M}}$ the set of all jump numbers and for every $j \in J_{\mathfrak{M}}$ denote by B_j the block of $\mathfrak{M}$ containing j. We say that an order $R^{\leq}_{\mathbf{A}}$ of ordered $\mathfrak{M}$-metric space $\mathbf{A}$ is* convex with respect to block equivalences *if every equivalence class of every $\sim_{B_j}$, $j \in J_S$, is an interval of $R^{\leq}_{\mathbf{A}}$. An ordered $\mathfrak{M}$-metric space which order is convex with respect to block equivalences is also called* convexly ordered $\mathfrak{M}$-metric space.

The following result is a direct analogy of Corollary 3.4 permitting jump numbers. We represent equivalence classes by closure vertices.

Lemma 3.13. *Let $\mathfrak{M}$ be finite distance monoid. Then the class $\overrightarrow{\mathcal{M}}_{\mathfrak{M}}$ of all convexly ordered $\mathfrak{M}$-metric spaces is a Ramsey class.*

Proof. We lift the language $L_{\mathfrak{M}}$ to $L'_{\mathfrak{M}}$ by adding the order $R^{\leq}$, unary relations R^{E_j} and binary relations R^{U_j}, $j \in J_{\mathfrak{M}}$. For a given ordered metric space $\mathbf{A} \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}}$ where the order is convex with respect to block equivalences denote by $L(\mathbf{A})$ the lift of $\mathbf{A}$ created by the following procedure:

(1) For every $j \in J_{\mathfrak{M}}$ enumerate equivalence classes of $\sim_{B_j}$ in $\mathbf{A}$ as $E^1_j, E^2_j, \ldots E^{n_j}_j$. Moreover order them in a way so for $v \in E^i_j$, $v' \in E^{i'}_j$ we have $v <_{\mathbf{A}} v'$ whenever $1 \leq i < i' \leq n_j$.
(2) For every $j \in J_{\mathfrak{M}}$ and $1 \leq j \leq n_j$ add a new vertex v^i_j and tuple (v^i_j) to $R^{E_j}_{L(\mathbf{A})}$.
(3) For every $j \in J_{\mathfrak{M}}$, $1 \leq i \leq n_j$ and $v \in E^i_j$ add tuple (v, v^i_j) to $R^{U_j}_{L(\mathbf{A})}$.

The linear order $R^{\leq}_{L(\mathbf{A})}$ extends the linear order of $\mathbf{A}$ by putting vertices v^i_j last in a way that $v^i_j <_{L(\mathbf{A})} v^{i'}_{j'}$ whenever $j < j'$ or $j = j'$ and $i < i'$.

The vertices in relations $R^{E_i}_{L(\mathbf{A})}$ represent individual equivalence classes and every vertex of $\mathbf{A}$ has an unique vertex in each of relations R^{E_j}, $j \in J_{\mathfrak{M}}$ connected to it by the closure tuple. We will call vertices of $\mathbf{A}$ the *original* vertices and vertices v^i_j the *closure* vertices.

Denote by $\overrightarrow{\mathcal{M}}'_{\mathfrak{M}}$ the class of all structures $L(\mathbf{A})$, $A \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}}$. Because for every ordered $\mathfrak{M}$-metric subspace $\mathbf{A}$ of convexly ordered $\mathfrak{M}$-metric space $\mathbf{B}$ we also know that $L(\mathbf{A})$ is a substructure of $L(\mathbf{B})$ the Ramsey property of $\overrightarrow{\mathcal{M}}'_{\mathfrak{M}}$ implies the Ramsey property of $\overrightarrow{\mathcal{M}}_{\mathfrak{M}}$. We show that $\overrightarrow{\mathcal{M}}'_{\mathfrak{M}}$ is an $(\mathcal{R}_{\mathfrak{M}}, \mathcal{U}_{\mathfrak{M}})$-multiamalgamation class where:

(1) The class $\mathcal{R}_{\mathfrak{M}}$ consists of all finite ordered $L'_{\mathfrak{M}}$ structures. This class is Ramsey by the application of Theorem **??**.
(2) The closure description $\mathcal{U}_{\mathfrak{M}}$ consists of pairs $(R^{U_j}, \mathfrak{M})$, $j \in J_{\mathfrak{M}}$, where $\mathfrak{M}$ is structure containing one vertex and no tuples.

It remains to verify the locally finite completion property (see Definition 2.6). We put $n = 4|\mathfrak{M}|$: Let $\mathbf{B} \in L(\mathfrak{M})$ and $\mathbf{C}'$ be an $\mathcal{U}_{\mathfrak{M}}$-closed structure with a homomorphism-embedding to $\mathbf{C}_0 \in \mathcal{R}_{\mathfrak{M}}$ such that every n-element substructure of $\mathbf{C}'$ has a completion in $\overrightarrow{\mathcal{M}}'_{\mathfrak{M}}$. Without loss of generality we may further assume that every vertex as well as every tuple in every relation of $\mathbf{C}'$ is contained in $\mathbf{B}$. We verify that $\mathbf{C}'$ has a strong completion in $\overrightarrow{\mathcal{M}}'_{\mathfrak{M}}$. We proceed in three steps:

(1) For every pair of original vertices $u, v \in \mathbf{C}'$ and jump number $j \in J_{\mathfrak{M}}$ such that there exists a walk from u to v where every distance is at most j there is c such that $(u,c),(v,c) \in R^{U_j}_{\mathbf{C}'}$ (that is, the u and v have the same $R^{U_j}_{\mathbf{C}'}$-closure):

   Let u' and w' be two neighbouring vertices in the walk. Because $(u',w') \in R^{d(u',w')}_{\mathbf{C}'}$ and every tuple of $\mathbf{C}'$ is part of a copy of $\mathbf{B}$ we know that there is a copy $\widetilde{B}$ in $\mathbf{C}'$ containing both u', w'. It follows that the unique vertex connected by $R^{U_j}_{\mathbf{C}'}$ to u' must be the same as the unique vertex connected by $R^{U_j}_{\mathbf{C}'}$ to v'. Consequently all vertices of the walk have the same $R^{U_j}_{\mathbf{C}'}$-closure.
(2) Denote by $\mathbf{G}$ the $\mathfrak{M}$-graph induced by $\mathbf{C}'$ on the set of original vertices. We show that there exists an $\mathfrak{M}$-metric space $\mathbf{G}'$ which is a strong completion of $\mathbf{G}$ (i.e. $\mathbf{G}$ is $\mathfrak{M}$-metric) and moreover every pair u and v of vertices of $\mathbf{C}'$ which have the same $R^{U_j}_{\mathbf{C}'}$-closure has distance in $\mathbf{G}'$ at most j.

   To simplify the discussion we assume that every pair of vertices u and v with same $R^{U_j}_{\mathbf{C}'}$-closure is connected by a walk where every distance is at most j. If that is not the case the distance of u and v in $\mathbf{G}$ can be set to j.

   To the contrary now assume that $\mathbf{G}$ is non-$\mathfrak{M}$-metric. By Proposition 3.2 there exists non $\mathfrak{M}$-metric cycle $\mathbf{K}$ in $\mathbf{G}$. Consider the family of unimportant paths in $\mathbf{K}$ given by Lemma 3.11. Let $\mathbf{P}$ be an unimportant path and j the smallest jump number of $\mathfrak{M}$ such that all distances in $\mathbf{P}$ are at most j. Then we know that there exists closure vertex $c \in C'$ such that $(u,c) \in R^{U_j}_{\mathbf{C}'}$ for every $u \in P$. We call c the *common closure of the path* $\mathbf{P}$.

   Create $\mathbf{K}'$ as the structure induced by $\mathbf{C}'$ on the set of all vertices of $\mathbf{K}$ which are not in unimportant paths, all initial and terminal vertices of unimportant path and the common closures of unimportant paths. By Lemma 3.11 there is no completion of $\mathbf{K}'$.

(3) Finally we complete the order of $\mathbf{C}'$ in a way that $\mathbf{G}'$ is ordered convexly. This can be done by first fixing the order of vertices in $R^{U_j}_{\mathbf{C}'}$ for j being the largest jump number and then proceed by second largest keeping the convex order and finally completing the order of original vertices.

It follows that $\mathbf{C} = L(\mathbf{G}')$ is the (strong) completion of $\mathbf{C}'$ in $\overrightarrow{\mathcal{M}}'_{\mathfrak{M}}$. □

Now we are ready to characterise $\mathfrak{M}$-metric Ramsey classes (extending Corollary 3.5):

Theorem 3.14 (Ramsey $\mathfrak{M}$-metric Spaces Theorem)**.** *Let $\mathfrak{M}$ be a dstance monoid. Then*
the class $\overrightarrow{\mathcal{M}}_{\mathfrak{M}}$ of all convexly ordered $\mathfrak{M}$-metric spaces is Ramsey.

Proof. □

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