Basic Probabilistic Method

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Introduction

Let's start with a simple problem:

Problem. In a party of 6 people, each pair are either friends or strangers. Prove that at least three of them are (pairwise) strangers or at least three of them are (pairwise) friends.

This problem is a special case for the *Graph Ramsey's Theorem.* And this workshop is not about this problem. Instead, we will see some basic probabilistic method techniques.

Useful stuff

The following bounds might be useful:

$$1 + x \le e^x \tag{1}$$

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k \tag{2}$$

Our probability spaces will be finite, which makes life (and also writing the following facts) much easier. This should be just a quick refresher of your basic probability and statistics class.

Fact 1 (Essential)**.** If some event has nonzero probability, then it can happen.

Definition 1 (Expectation)**.** Let X be a random variable. Then we define the expected value of X as

$$\mathbb{E}\,X = \sum x \Pr[X = x].$$

Fact 2 (Linearity of expectation)**.** For X and Y random variables (on the same probability space) it is true that $\mathbb{E}[X+Y] = \mathbb{E}\,X + \mathbb{E}\,Y$ and $\mathbb{E}[\alpha X] = \alpha\,\mathbb{E}\,X$ for any $\alpha \in \mathbb{R}$.

Fact 3 (Essential, too)**.** If X is a random variable, then there is a point in the probability space with $X \ge \mathbb{E}\,X$ and with $X \le \mathbb{E}\,X$.

Fact 4 (Pitcher lemma)**.** If the probability for a pitcher to break is p, then, in expectation, it needs to go the the well $\frac{1}{p}$ times to break.

Problems

Try to solve these problems. There are hints for them at the end of the handout. They are in almost no specific order, but I can assure you that the problems numbered by primes as well as squares or the problems with numbers whose Euler totient function is a power of two, are fun. The last two problems are slightly more difficult, but don't be intimidated!

Problem 1. Let A be a set. Prove that for any $n \ge 2$ and any family $\mathcal{F} \subset \binom{A}{n}$ (subsets of A of size n) with $|\mathcal{F}| < 2^{n-1}$ there is a 2-colouring of A such that no set in $\mathcal{F}$ is monochromatic.

Problem 2. Let A be a set. Prove that for any $n \ge 4$ and any family $\mathcal{F} \subset \binom{A}{n}$ with $|\mathcal{F}| < \frac{4^{n-1}}{3^n}$ there is a 4-colouring of A such that every set in $\mathcal{F}$ is proud (i.e. contains items of all four colours).

Problem 3. Let G be a graph with e edges. Prove that G contains a bipartite (non-induced) subgraph with at least $\frac{e}{2}$ edges. Can you prove that also deterministically? (But first solve it using randomness.)

Problem 4 (Lower bound for Ramsey theorem)**.** Prove that $R(k,k) > \lfloor 2^{k/2} \rfloor$ for all large enough[1] k, i.e. that there exists a 2-colouring of edges of $K_{\lfloor 2^{k/2} \rfloor}$ which contains no monochromatic K_k as a subgraph.

Problem 5. Let $G = (V,E)$ be a graph with n vertices and minimum degree δ. Prove that there is a set $D \subset V$ with $|D| \le n \cdot \frac{1+\ln(\delta+1)}{\delta+1}$, such that every vertex either is in D or has a neighbour in D. (D is called a *dominating set.*)

Problem 6. Jarda wants to get quarters from all possible states. There are n states in the USA and, for simplicity, when Jarda is getting change at the Busch Dining Hall, quarters from each state are equally likely. What is the expected number of lunches Jarda needs to buy to get a full collection (Jarda always gets exactly one quarter).

Problem 7 (Erdös-Ko-Rado theorem)**.** Let $n \ge 2k$ be positive integers and let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a family of subsets such that for any $A, B \in \mathcal{F}$ we have $A \cap B \neq \emptyset$. Prove that $|\mathcal{F}| \le \binom{n-1}{k-1}$.

Problem 8. Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be vectors with $|v_i| = 1$. Show that there exist $\varepsilon_1, \ldots, \varepsilon_n \in \pm 1$ such that $|\sum \varepsilon_i v_i| \ge \sqrt{n}$.

Problem 9. Let G be an n-vertex bipartite graph. Further let each vertex be assigned a list of $> \log_2 n$ possible colours. Show that it is possible to give each vertex a colour from its list such that no two adjacent vertices have the same colour.

Problem 10. Prove that any graph with n vertices and nd edges ($d \ge \frac{1}{2}$) contains an independent set of size at least $\frac{n}{4d}$.

Problem 11. [Kraft inequality] Let $\mathfrak{F}$ be a finite collection of binary strings of finite lengths, such that no member of $\mathfrak{F}$ is a prefix of another one. Let N_i denote the number of strings of length i in $\mathfrak{F}$. Prove that $\sum_i \frac{N_i}{2^i} \le 1$.

Problem 12. [Crossing lemma] Let G be a graph with n vertices and $e \ge 4n$ edges. Then in any drawing of G in a plane, there will be at least $\frac{e^3}{64n^2}$ pairs of crossing edges.

Problem 13. Let an $n \times n$ array of lightbulbs be given, each either on or off. For each row and column there is a switch that changes the state of all lightbulbs in the given row/column. Prove that for any initial configuration it is possible to perform switching in such a way that there wil be at least $\left(\sqrt{\frac{2}{\pi}} + o(1)\right) n^{3/2}$ more lightbulbs on than off. You can use the following estimate:

$$2^{1-n} \sum_{i=1}^{\lfloor n/2 \rfloor} (n-2i)\binom{n}{i} = \left(\sqrt{\frac{2}{\pi}} + o(1)\right)\sqrt{n}.$$

Acknowledgements

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[1] so that you can be generous with estimates, $k \ge 4$ worked for me

Hints

Yayyy, hints!

Hint (1)**.** Pick a random colouring. What is the probability, that some $f \in \mathcal{F}$ is monochromatic? What is the expected number of monochromatic sets in $\mathcal{F}$?

Hint (2)**.** Did you solve Problem 1?

Hint (3)**.** Pick a random set of vertices. What is the expected number of edges going out from this set? For deterministic approach, try greedy.

Hint (4)**.** Did you try a random colouring? Prove that $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ for $n = \lfloor 2^{k/2} \rfloor$.

Hint (5)**.** Let A be a random subset of vertices, where each vertex is with probability p. Then let B the set of vertices, which are not in A and neither have a neighbour in A. What is the expected size of $A \cup B$?

Hint (6)**.** Let Q_i be the random variable denoting the number of trials in order to get i-th new quarter given that Jarda already has $i-1$ different quarters. What is $\mathbb{E}\, Q_i$?

Hint (7)**.** I am pretty sure I showed this problem on the board.

Hint (8)**.** Square both sides, write the left side as a dot product, expand and use linearity of expectation.

Hint (9)**.** Split the colours randomly into two sets.

Hint (10)**.** Pick each vertex randomly with some probability p and then throw away one for each edge that you got in your subgraph. What is a good value for p?

Hint (11)**.** What does the expression $\sum_i \frac{N_i}{2^i}$ count?

Hint (12)**.** First get rid of multiple crossings of two edges. Then prove that in any such drawing of any graph H there are at least $|E(H)| - 3|V(H)|$ pairs of edges crossing. Finally take a random vertex subset of G independently with probability p and look at what it induces. Can you use the weak estimate $(e - 3v)$ for the random subset to get something stronger for G?

Hint (13)**.** This is a fun problem. Don't give up!