Ramsey expansions of metrically homogeneous graphs

Andrés Aranda, David Bradley-Williams[a], Jan Hubička[b,1], Miltiadis Karamanlis[c], Michael Kompatscher[d,2], Matěj Konečný[e], Micheal Pawliuk[f]

[a] *Computer Science Institute of Charles University (IUUK)*
Charles University
Prague, Czech Republic
[b] *Computer Science Institute*
Charles University
Prague, Czech Republic
[c] *Department of Mathematics*
National Technical University of Athens
Athens, Greece
[d] *Institut für Computersprachen*
Theory and Logic Group
Technische Universität Wien
Wien, Austria
[e] *Charles University*
Prague, Czech Republic
[f] *Department of Mathematics and Statistics*
University of Calgary
Calgary, Canada

Abstract

We discuss the Ramsey property, the existence of a stationary independence relation and the coherent extension property for partial isometries (coherent EPPA) for all classes of metrically homogeneous graphs from Cherlin's catalogue, which is conjectured to include all such structures. We show that, with the exception of tree-like graphs, all metric spaces in the catalogue have precompact Ramsey expansions (or lifts) with the expansion property. With one exception we can also characterise the existence of a stationary independence relation and the coherent EPPA.

Our results can be seen as a new contribution to Nešetřil's classification programme of Ramsey classes and as empirical evidence of the recent convergence in techniques employed to establish the Ramsey property, the expansion (or lift or ordering) property, EPPA and the existence of a stationary independence relation. At the heart of our proof is a canonical way of completing edge-labeled graphs to metric spaces in Cherlin's classes. The existence of such a "completion algorithm" then allows us to apply several strong results in the areas that imply EPPA and respectively the Ramsey property.

The main results have numerous corollaries on the automorphism groups of the Fraïssé limits of the classes, such as amenability, unique ergodicity, existence of universal minimal flows, ample generics, small index property, 21-Bergman property and Serre's property (FA).

Email addresses: `andres.aranda@gmail.com` (Andrés Aranda), `David.Bradley-Williams@uni-duesseldorf.de` (David Bradley-Williams), `hubicka@iuuk.mff.cuni.cz` (Jan Hubička), `michael@logic.at` (Miltiadis Karamanlis), `michael@logic.at` (Michael Kompatscher), `matejkon@gmail.com` (Matěj Konečný), `mpawliuk@ucalgary.ca` (Micheal Pawliuk)

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Contents

1. Introduction

A (countable) structure is called *homogeneous* if every isomorphism between finite substructures is induced by an automorphism. A *metrically homogeneous graph* is a connected graph with the property that the associated metric space is a homogeneous metric space. The *associated metric space* of a graph has the vertices as points, and the distance between two points is the length of the shortest path connecting them.

As a new contribution to the well known classification programme of homogeneous structures (of Lachlan and Cherlin [1, 2, 3]), Cherlin [4] has recently provided a catalogue of metrically homogeneous graphs which is conjectured to be complete.

This paper is concerned about properties of ages of structures from Cherlin's catalogue. To state our main results we introduce some terminology first. A graph is *bipartite* if it contains no odd cycles. The *diameter* of a graph is the maximal distance in its associated metric space. A graph of diameter δ is *antipodal* if in its associated metric space for every vertex there exists at most one vertex in distance δ. A graph is *tree-like* if it is isomorphic to a graph $T_{m,n}$, $2 \leq m, n \leq \infty$, defined as follows:

Definition 1.1. For $2 \leq m, n \leq \infty$ the graph $T_{m,n}$ is a regular tree-like graph in which the blocks (two-connected components) are cliques of order n and every vertex is a cut vertex, lying in precisely m blocks.

Our main results can be summarised as follows (all the remaining notions will be introduced below):

Theorem 1.1. *Let* Γ *be the associated metric space of a countably infinite metrically homogeneous graph* $\mathbf{G}$ *of diameter* δ *from Cherlin's catalogue. Then one of the following applies:*

1. *If* $\mathbf{G}$ *is a tree-like graph, then* $\mathrm{Age}(\Gamma)$ *has no precompact Ramsey expansion.*
2. *If* $\mathbf{G}$ *is not tree-like, then one of the following holds:*
   - (a) *If* $\mathbf{G}$ *is antipodal, denote by* $\mathcal{A}$ *the subclass of* $\mathrm{Age}(\Gamma)$ *of such metric spaces where for every vertex there exists a unique other vertex in distance* δ*. Then:*
     - i. *If* $\mathbf{G}$ *is not bipartite or has odd diameter, then* $\mathcal{A}$ *is Ramsey when extended by a linear ordering.*
     - ii. *If* $\mathbf{G}$ *is bipartite of even diameter, then* $\mathcal{A}$ *is Ramsey when extended by a unary predicate denoting the bipartition and convex linear ordering.*
   - (b) *If* $\mathbf{G}$ *is not antipodal then:*
     - i. *If* $\mathbf{G}$ *is not bipartite, then the class of free orderings of* $\mathrm{Age}(\Gamma)$ *is Ramsey.*
     - ii. *If* $\mathbf{G}$ *is bipartite, then* $\mathrm{Age}(\Gamma)$ *is Ramsey when extended by a unary predicate denoting the bipartition and convex linear orderings .*

By "a predicate denoting the bipartition" we mean a unary predicate defining two equivalence classes of vertices, such that all distances in the same equivalence class are even.

All the described Ramsey expansions have the expansion property.

We thus completely characterise the Ramsey expansions of all currently known metrically homogeneous graphs. In addition to that we will show two related properties, arriving at an almost complete characterisation and an open problem.

Theorem 1.2. *Let* Γ *be the associated metric space of a countably infinite metrically homogeneous graph* $\mathbf{G}$ *from Cherlin's catalogue.*

1. *If* $\mathbf{G}$ *is a tree-like graph, then* $\mathrm{Age}(\Gamma)$ *does not have EPPA.*
2. *If* $\mathbf{G}$ *is not tree-like then:*
   - (a) *If* $\mathbf{G}$ *is antipodal, and:*
     - i. *If* $\mathbf{G}$ *is non-bipartite of even diameter or bipartite of odd diameter, then* $\mathrm{Age}(\Gamma)$ *has coherent EPPA.*

ii. *If* $\mathbf{G}$ *is bipartite of even diameter or non-bipartite of odd diameter, then* $\mathrm{Age}(\Gamma)$ *extended by a unary predicate denoting podality has coherent EPPA.*

(b) *If* $\mathbf{G}$ *is not antipodal then it has coherent EPPA.*

By "a predicate denoting podality" we mean a unary predicate defining two equivalence classes of vertices, such that every there is no pair of vertices in distance δ *are in the same equivalence class.*

We do not know if the statement of Theorem 1.2 about antipodal metric space is best possible. The following question remains open:

Problem 1.3. *Do the ages of the metric spaces associated with antipodal metrically homogeneous graphs of odd diameter and/or antipodal bipartite metrically homogeneous graphs of even diameter have EPPA?*

This problem will be discussed in additional detail in Section 10.

Theorem 1.4. *Let* Γ *be the associated metric space of a countably infinite metrically homogeneous graph* $\mathbf{G}$ *from Cherlin's catalogue.*

1. *If* $\mathbf{G}$ *is isomorphic to* $T_{m,n}$ *then* Γ *has no stationary independence relation and it a has a local stationary independence relation if and only if* $m = \infty$ *and* $n \in \{2, 3, \infty\}$.
2. *If* $\mathbf{G}$ *is not tree-like then:*
   (a) *If* $\mathbf{G}$ *is antipodal, and:*
      i. *If* $\mathbf{G}$ *is not bipartite and has even diameter, then there exists a stationary independence relation on* Γ.
      ii. *If* $\mathbf{G}$ *is bipartite and has odd diameter, then there exists a local stationary independence relation, but there is no stationary independence relation on* Γ.
      iii. *Otherwise there is no local stationary independence relation on* Γ.
   (b) *If* $\mathbf{G}$ *is not antipodal:*
      i. *If* $\mathbf{G}$ *is not bipartite, then there exists a stationary independence relation on* Γ.
      ii. *If* $\mathbf{G}$ *is bipartite, then there exists a local stationary independence relation, but there is no stationary independence relation on* Γ.

1.1. Edge-labelled Graphs and metric completions

Our arguments are based on analysis of an algorithm to fill gaps in incomplete metric spaces presented in Section 4. This algorithm works for a significant (primitive) part of Cherlin's catalogue and leads to a strong notion of "canonical amalgamation." When constraints on the amalgamation classes are pushed to extremes, new phenomena (such as antipodality or bipartiteness) appear. We show that even in this case our algorithm is useful, provided that we apply some additional techniques. First, we introduce the necessary notation to speak about metric spaces with gaps.

Definition 1.2. An *edge-labelled graph* $\mathbf{G}$ is a pair (G, d) where G is the *vertex set* and d is a partial function from G^2 to $\mathbb{N}$ such that $d(u, v) = 0$ if and only if $u = v$, and $d(u, v) = d(v, u)$ whenever either number is defined. A pair of vertices u, v on which $d(u, v)$ is defined is called an *edge* of $\mathbf{G}$. We also call $d(u, v)$ the *length of the edge* u, v.

The standard graph-theoretic notions of homomorphism, embedding, and isomorphism extend naturally to edge-labelled graphs. Our subgraphs will always be induced (details are given in Section 2). An edge-labelled graph can also be seen as a relational structure (as discussed in Section 2.1). We find it convenient to use notation that resembles the standard notation of metric spaces.

We denote by $\mathcal{G}^\infty$ the class of all finite edge-labelled graphs and by $\mathcal{G}^\delta$ the subclass of $\mathcal{G}^\infty$ of those graphs containing no edge of length greater than δ.

A (edge-labeled) graph $\mathbf{G}$ is *complete* if every pair of vertices forms an edge; a complete edge-labelled graph $\mathbf{G}$ is called a *metric space* if the *triangle inequality* holds, that is $d(u, w) \le d(u, v) + d(v, w)$ for every

$u, v, w \in G$. An edge-labelled graph $\mathbf{G} = (G, d)$ is *metric* if there exists a metric space $\mathbf{M} = (G, d')$ such that $d(u, v) = d'(u, v)$ for every edge u, v of $\mathbf{G}$. Such a metric space $\mathbf{M}$ is also called a *(strong) metric completion* of $\mathbf{G}$.

Given $\mathbf{G} = (G, d) \in \mathcal{G}^\infty$ the *path distance $d^+(u, v)$ of u and v* is the minimum

$$\ell = \sum_{1 \leq i \leq n-1} d(u_i, u_{i+1})$$

taken over all possible sequences of vertices for which $u_1 = u, u_2, \dots u_n = v$ and $d(u_i, u_{i+1})$ is defined for every $i \leq n - 1$. If there is no such sequence we put $\ell = \infty$. It is a well-known fact that a connected edge-labelled graph $\mathbf{G} = (G, d)$ is metric if and only if $d(u, v) = d^+(u, v)$ for every edge of $\mathbf{G}$. In this case (G, d^+) is a metric completion of $\mathbf{G}$ which we refer to as the *shortest path completion*. This completion algorithm also leads to an easy characterisation of metric graphs: $\mathbf{G}$ is metric if and only if it does not contain a *non-metric cycle*, by which we mean a graph edge-labelled cycle such that one distance in the cycle is greater than sum of the remaining distances. See e.g. [5] for details.

Our algorithm is a generalisation of the shortest path completion algorithm, tailored to preserve some other properties (e.g., forbidding long cycles while shortest path completion permits forbidding only short cycles).

1.2. Extension property for partial automorphisms (EPPA)

A *partial automorphism* of the structure $\mathbf{A}$ is an isomorphism $f : \mathbf{B} \to \mathbf{B}'$ where $\mathbf{B}$ and $\mathbf{B}'$ are substructures of $\mathbf{A}$. We say that a class of finite structures $\mathcal{K}$ has the *extension property for partial automorphisms* (*EPPA*, sometimes called the *Hrushovski extension property* or in the context of metric spaces the *extension property for partial isometries*) if for all $\mathbf{A} \in \mathcal{K}$ there is $\mathbf{B} \in \mathcal{K}$ such that $\mathbf{A}$ is a substructure of $\mathbf{B}$ and every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$. We call $\mathbf{B}$ with such property an *EPPA extension of* $\mathbf{A}$

In addition to being a non-trivial and beautiful combinatorial property, classes with EPPA have further interesting properties. For example, Kechris and Rosendal [6] have shown that the automorphism groups of their Fraïssé limits are amenable.

In 1992, Hrushovski [7] showed that the class $\mathcal{G}$ of all finite graphs has EPPA. A combinatorial argument for Hrushovski's result was given by Herwig and Lascar [8] along with a non-trivial strengthening for certain, more restricted, classes of structures described by forbidden homomorphisms. This result was independently used by Solecki [9] and Vershik [10] to prove EPPA for the class of all finite metric spaces with integer, rational or real distances. (Because EPPA is a property of finite structures, there is no difference between integer and rational distances. The techniques also naturally extend to real numbers. For our presentation we will consider integer distances only.) These results were further strengthened by Rosendal [11, 12]. Recently Conant developed this argument to generalised metric spaces [13], where the distances are elements of a distance monoid. As a special case this implies EPPA for classes of metric spaces limited to a given subset of integers and to metric graphs omitting short triangles of odd perimeter.

Motivated by results on ample generics by Hodges, Hodkinson, Lascar and Shelah [14], Siniora and Solecki [15, 16] introduced the stronger notion of *coherent EPPA* (where the extensions to automorphisms compose whenever the partial automorphisms do, see Definition 2.8) along with the amalgamation property with automorphisms (where the automorphism group of the amalgamation is the same as of the free amalgamation, see Section 9.1).

1.3. Ramsey classes

The notion of Ramsey classes was isolated in the 1970s and, being a strong combinatorial property, it has found numerous applications, for example, in topological dynamics [17]. It was independently proved by Nešetřil and Rödl [18] and Abramson-Harrington [19] that the class of all finite linearly ordered hypergraphs is a Ramsey class. Several new classes followed. We briefly outline the results related to Ramsey classes of metric spaces, see [20, 21, 22, 5] for further references.

For structures $\mathbf{A}, \mathbf{B}$ denote the set of all substructures of $\mathbf{B}$ that are isomorphic to $\mathbf{A}$ by $\binom{\mathbf{B}}{\mathbf{A}}$.

Definition 1.3. A class $\mathcal{C}$ of structures is a *Ramsey class* if for every two objects $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{C}$ and for every positive integer k there exists a structure $\mathbf{C}$ in $\mathcal{C}$ such that the following holds: For every partition of $\binom{\mathbf{C}}{\mathbf{A}}$ into k classes there exists an $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ is contained in a single class of the partition. For short we then write

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_k.$$

It was observed by Nešetřil [23] that under mild assumptions every Ramsey class is an amalgamation class. The Nešetřil classification programme of Ramsey classes [24] asks which amalgamation classes provided by the classification programme of homogeneous structures yield a Ramsey class. Amalgamation classes are often not Ramsey for simple reasons, such as the lack of a linear order on the vertices of its structures (which is known to be present in every Ramsey class ([6], see e.g. [22, Proposition 2.22]). For this reason we consider enriched classes, where the language of an amalgamation class is expanded by additional relations. We refer to these classes as *Ramsey expansions* (or *lifts*), see Section 2.2.

In 2005 Nešetřil [25] showed that the class of all finite metric spaces is a Ramsey class when enriched by free linear ordering of the vertices. This result was extended to some subclasses $\mathcal{A}_S$ of finite metric spaces where all distances belong to a given set S by Nguyen Van Thé [26]. Recently Hubička and Nešetřil further generalised this result to $\mathcal{A}_S$ for all feasible choices of S [5], earlier identified by Sauer [27], as well as to the class of metric spaces omitting cycles of odd perimeter. One additional case is discussed by Sokić [**?**].

1.4. Stationary independence relations

A stationary independence relation (Definition 1.4) is a generalisation of the Katětov constructions recently developed by Tent and Ziegler [28] to show the simplicity of the automorphism group of the Urysohn space. Müller [29] used the same property to prove the universality of the same automorphism group. Example of structures with the stationary independence relation include free amalgamation classes, metric spaces [28] and the Hrushovski predimension construction (given by Evans, Ghadernezhad and Tent [30]).

Given a structure $\mathbf{M}$ and finite substructures $\mathbf{A}$ and $\mathbf{B}$, the substructure generated by their union is denoted by $\langle\mathbf{AB}\rangle$. A stationary independence relation is a ternary relation on finite substructures of a given homogeneous structure $\mathbf{M}$. The substructures $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are in the relation, roughly speaking, when $\mathbf{M}$ induces a "canonical amalgamation" of $\langle\mathbf{AC}\rangle$ and $\langle\mathbf{BC}\rangle$ over $\mathbf{C}$. In this case we write $\mathbf{A} \mathop{\smile\!\!\!\!\!\!|}_{\mathbf{C}} \mathbf{B}$.

Stationary independence relations are axiomatised as follows:

Definition 1.4 ((Local) Stationary Independence Relation)**.** Assume $\mathbf{M}$ to be a homogeneous structure. A ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ on the finite substructures of $\mathbf{M}$ is called a *stationary independence relation* (*SIR*) if the following conditions are satisfied:

SIR1 (Invariance). The independence of finitely generated substructures in $\mathbf{M}$ only depends on their type. In particular, for any automorphism f of $\mathbf{M}$, we have $\mathbf{A} \mathop{\smile\!\!\!\!\!\!|}_{\mathbf{C}} \mathbf{B}$ if and only if $f(\mathbf{A}) \mathop{\smile\!\!\!\!\!\!|}_{f(\mathbf{C})} f(\mathbf{B})$.

SIR2 (Symmetry). If $\mathbf{A} \mathop{\smile\!\!\!\!\!\!|}_{\mathbf{C}} \mathbf{B}$, then $\mathbf{B} \mathop{\smile\!\!\!\!\!\!|}_{\mathbf{C}} \mathbf{A}$.

SIR3 (Monotonicity). If $\mathbf{A} \mathop{\smile\!\!\!\!\!\!|}_{\mathbf{C}} \langle\mathbf{BD}\rangle$, then $\mathbf{A} \mathop{\smile\!\!\!\!\!\!|}_{\mathbf{C}} \mathbf{B}$ and $\mathbf{A} \mathop{\smile\!\!\!\!\!\!|}_{\langle\mathbf{BC}\rangle} \mathbf{D}$.

SIR4 (Existence). For any $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ in $\mathbf{M}$, there is some $\mathbf{A}' \models \mathrm{tp}(\mathbf{A}/\mathbf{C})$ with $\mathbf{A}' \mathop{\smile\!\!\!\!\!\!|}_{\mathbf{C}} \mathbf{B}$.

SIR5 (Stationarity). If $\mathbf{A}$ and $\mathbf{A}'$ have the same type over $\mathbf{C}$ and are both independent over $\mathbf{C}$ from some set $\mathbf{B}$, then they also have the same type over $\langle\mathbf{BC}\rangle$.

If the relation $\mathbf{A} \mathop{\smile\!\!\!\!\!\!|}_{\mathbf{C}} \mathbf{B}$ is only defined for nonempty $\mathbf{C}$, we call $\mathop{\smile\!\!\!\!\!\!|}$ a *local* stationary independence relation.

Here $\mathrm{tp}(\mathbf{A}/\mathbf{C})$ denotes the type of $\mathbf{A}$ over $\mathbf{C}$, see Definition 2.1.

1.5. *Obstacles to completion*

The list of subclasses of metric spaces with Ramsey expansions corresponds closely to the list of classes with EPPA. The similarity of these results is not a coincidence. All of the proofs proceed from a given metric space and, by a non-trivial construction, build an edge-labelled graph with either the desired Ramsey property or EPPA. However those edge-labelled graphs might not be complete. Using some detailed information about when and how they can be completed to the given class of metric spaces $\mathcal{A}$, it is then possible to find an actual witness for the Ramsey property or EPPA in $\mathcal{A}$. The actual "amalgamation engines" have been isolated (see Theorems 2.1 and 2.2) and are based on characterising each class by a set of obstacles in the sense of the definition below. Given a set $\mathcal{O}$ of edge-labelled graphs, let $\mathrm{Forb}(\mathcal{O})$ denote the class of all finite edge-labelled graphs $\mathbf{G}$ such that there is no $\mathbf{O} \in \mathcal{O}$ with a homomorphism $\mathbf{O} \to \mathbf{G}$.

Definition 1.5. Given a class of metric spaces $\mathcal{A}$, we say that $\mathcal{O}$ is the *set of obstacles of* $\mathcal{A}$ if $\mathcal{A} \subseteq \mathrm{Forb}(\mathcal{O})$ and moreover every $\mathbf{G} \in \mathrm{Forb}(\mathcal{O})$ has a metric completion into $\mathcal{A}$.

1.6. *Outline of the paper*

Cherlin's catalogue, given in Section 3, provides a rich spectrum of structures. We follow the catalogue in an order corresponding to the proof techniques (or main properties of the amalgamation classes), rather than strictly following the order of the catalogue as presented by Cherlin [4]. Several classes are refined and individual special cases are considered separately.

A *triangle* is a triple of distinct vertices $u, v, w \in M$ and its *perimeter* is $d(u,v)+d(v,w)+d(w,u)$. Classes described by constraints on triangles form an essential part of the catalogue; we call them *3-constrained classes*. These classes can be described by means of five numerical parameters:

Definition 1.6 (Triangle constraints)**.** Given integers δ, K_1, K_2, C_0 and C_1 we consider the class $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ of all finite metric spaces $\mathbf{M} = (M, d)$ with integer distances such that $d(u, v) \leq \delta$ (we call δ the *diameter* of $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$) for every $u, v \in M$ and for every triangle $u, v, w \in M$ with perimeter $p = d(u, v) + d(u, w) + d(v, w)$ where the following are true:

- if p is odd then $2K_1 < p < 2K_2 + 2m$,
- if p is odd then $p < C_1$, and
- if p is even then $p < C_0$,

where $m = \{\min(d(u, v), d(u, w), d(v, w)\}$ is the length of its shortest edge.

Intuitively, the parameter K_1 forbids all odd cycles shorter than $2K_1 + 1$, while K_2 ensures that the difference in length between even- and odd-distance paths connecting any pair of vertices is less than $2K_2 + 1$. The parameters C_0 and C_1 forbid induced long even and odd cycles respectively. Not every combination of numerical parameters makes sense and leads to an amalgamation class. Those that do are characterised by Cherlin's Admissibility Theorem 3.2.

We consider the following classes:

Spaces of diameter 2: Classes with diameter 2 are not discussed at all in this paper, because all the relevant results have already been established; these are just the homogeneous graphs. The Ramsey properties of homogeneous graphs were first studied by Nešetřil [23]. See also [31], which states the relevant results in more modern language and also shows the ordering properties in all classes from the Lachlan-Woodrow catalogue (of homogeneous graphs) [2]. The extension property for partial automorphism of the class of all finite graphs was shown by Hrushovski [7] and the remaining cases are particularly easy.

Primitive 3-constrained spaces: A metrically homogeneous graph $\mathbf{G}$ is *primitive* if there are no non-trivial $\mathrm{Aut}(\mathbf{G})$-invariant equivalence relations on its vertices.

In Section 4 we consider the richest regular family of amalgamation classes in the catalogue. This case contains the class of all finite metric spaces of finite diameter δ and, more generally, most classes $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$, $\delta < \infty$, with parameters satisfying Case (II) or (III) of Theorem 3.2.

These classes are closed under strong amalgamation. We describe a generalised completion algorithm which allows us to show the coherent EPPA and present a Ramsey expansion.

Bipartite 3-constrained spaces: In Section 6 we consider the 3-constrained spaces $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$, $\delta < \infty$ that contain no odd triangles, but triangles with edge length $\delta, \delta, 2$ are allowed (and thus $C_0 > 2\delta+2$). This corresponds to the non-antipodal Case (I) and is similar to primitive 3-constrained spaces where the bipartitions form definable equivalence relations and thus introduce imaginary elements. The existence of imaginaries has some consequences on the completion algorithm, Ramsey expansions, and coherent EPPA.

Spaces with Henson constraints: As discussed in Section 5, many 3-constrained classes can be further restricted by Henson constraints (spaces with distances 1 and δ only). These constraints cannot be represented in the form of forbidden triangles. We show that they have little effect on our constructions because the completion algorithm never introduces new Henson constrains in any class where Henson constrains are admissible (Definition 3.3).

Antipodal spaces: Here we consider 3-constrained classes where edges of length δ form a matching. Every amalgamation class closed under forming antipodal companions (see Definition 7.2) can be extended to an antipodal metric space and these are special cases of our constructions because the matching implies non-trivial algebraic closure in the Fraïssé limits (in other words the class is not closed for strong amalgamation). We show how to carry the Ramsey and EPPA results from the underlying class to its antipodal variant in Section 7 and also discuss antipodal Henson constraints.

Classes with infinite diameter: So far we have only discussed classes of finite diameter. However we can derive EPPA and the Ramsey property in the remaining infinite diameter cases from what we already know from the classes with finite diameter, as we will show in Section 8.1. In Section 8.2, we deal with ages of tree-like graphs $T_{m,n}$. These graphs generalise $\mathbb{Z}$ seen as a metric space, and we show that their basic properties are similar—there is no precompact Ramsey lift and no EPPA. Both conclusions follow from the fact that the algebraic closure of any vertex is infinite.

2. Preliminaries

We first review the standard model-theoretic notions of relational structures and amalgamation classes (see, for example [32]). Next, we introduce the relevant "amalgamation engines" used to build Ramsey objects and EPPA extensions throughout this paper.

A language L is a set of relational symbols $R \in L$, each associated with a natural number $a(R)$ called *arity*. A *(relational) L-structure* $\mathbf{A}$ is a pair $(A, (R_{\mathbf{A}}; R \in L))$ where $R_{\mathbf{A}} \subseteq A^{a(R)}$ (i.e. $R_{\mathbf{A}}$ is a $a(R)$-ary relation on A). The set A is called the *vertex set* or the *domain* of $\mathbf{A}$ and elements of A are *vertices*. The language is usually fixed and understood from the context (and it is in most cases denoted by L). If A is a finite set, we call $\mathbf{A}$ a *finite structure*. We consider only structures with countably many vertices. The class of all (countable) relational L-structures will be denoted by $\mathrm{Rel}(L)$.

A *homomorphism* $f : \mathbf{A} \to \mathbf{B} = (B, (R_{\mathbf{B}}; R \in L))$ is a mapping $f : A \to B$ satisfying for every $R \in L$ the implication $(x_1, x_2, \ldots, x_{a(R)}) \in R_{\mathbf{A}} \implies (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in R_{\mathbf{B}}$. (For a subset $A' \subseteq A$ we denote by $f(A')$ the set $\{f(x) : x \in A'\}$ and by $f(\mathbf{A})$ the homomorphic image of a structure.) If f is injective, then f is called a *monomorphism*. A monomorphism is called *embedding* if the above implication is an equivalence, i.e. if for every $R \in L$ we have $(x_1, x_2, \ldots, x_{a(R)}) \in R_{\mathbf{A}} \iff (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in R_{\mathbf{B}}$. If f is an embedding which is an inclusion then $\mathbf{A}$ is a *substructure* (or *subobject*) of $\mathbf{B}$. By the *age of a structure* $\mathbf{A}$, or $\mathrm{Age}(\mathbf{A})$ we denote the class of all finite substructures of $\mathbf{A}$. For an embedding $f : \mathbf{A} \to \mathbf{B}$ we say that $\mathbf{A}$ is *isomorphic* to $f(\mathbf{A})$ and $f(\mathbf{A})$ is also called a *copy* of $\mathbf{A}$ in $\mathbf{B}$. We define $\binom{\mathbf{B}}{\mathbf{A}}$ as the set of all copies of $\mathbf{A}$ in $\mathbf{B}$.

Let $\mathbf{A}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ be relational structures and α_1 an embedding of $\mathbf{A}$ into $\mathbf{B}_1$, α_2 an embedding of $\mathbf{A}$ into $\mathbf{B}_2$, then every structure $\mathbf{C}$ with embeddings $\beta_1 : \mathbf{B}_1 \to \mathbf{C}$ and $\beta_2 : \mathbf{B}_2 \to \mathbf{C}$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ is called an *amalgamation* of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2. See Figure 1. We will call $\mathbf{C}$ simply an *amalgamation* of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ (as in most cases α_1 and α_2 can be chosen to be inclusion embeddings).

We say that an amalgamation is *strong* when $\beta_1(x_1) = \beta_2(x_2)$ if and only if $x_1 \in \alpha_1(A)$ and $x_2 \in \alpha_2(A)$. Less formally, a strong amalgamation glues together $\mathbf{B}_1$ and $\mathbf{B}_2$ with an overlap no greater than the copy of

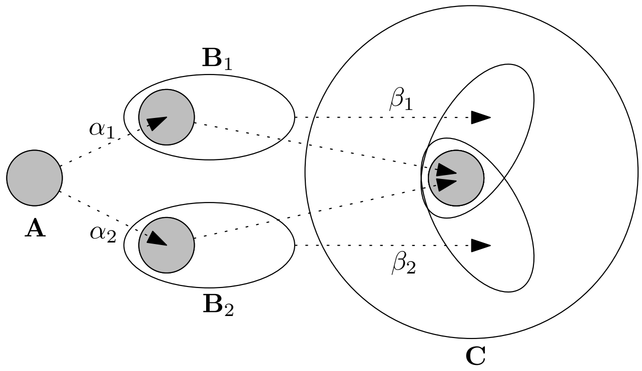

Figure 1: An amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$.

$\mathbf{A}$ itself. A strong amalgamation is *free* if there are no tuples in any relations of $\mathbf{C}$ spanning vertices from both $\beta_1(B_1 \setminus \alpha_1(A))$ and $\beta_2(B_2 \setminus \alpha_2(A))$.

An *amalgamation class* is a class $\mathcal{K}$ of finite structures satisfying the following three conditions:

Hereditary property: For every $\mathbf{A} \in \mathcal{K}$ and a substructure $\mathbf{B}$ of $\mathbf{A}$ we have $\mathbf{B} \in \mathcal{K}$;

Joint embedding property: For every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{C}$ contains both $\mathbf{A}$ and $\mathbf{B}$ as substructures;

Amalgamation property: For $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and α_1 an embedding of $\mathbf{A}$ into $\mathbf{B}_1$, α_2 an embedding of $\mathbf{A}$ into $\mathbf{B}_2$, there is a $\mathbf{C} \in \mathcal{K}$ which is an amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2.

Definition 2.1 (Type)**.** Let $L(\mathbf{X})$ denote the language $L \cup \{c_x : x \in X\}$, where each c_x is a constant symbol interpreted as the vertex x. By a *type over* $\mathbf{X}$, we mean a maximal satisfiable set of $L(\mathbf{X})$ formulas $p(x)$ with free variables x. The type $\mathrm{tp}(a/\mathbf{X})$ of a tuple a over $\mathbf{X}$ is the set of all $L(\mathbf{X})$ formulas which are satisfied by a.

2.1. Edge-labelled graphs as relational structures

For notational convenience we introduced the edge-lablled graphs (Definition 1.2) and we will view them, equivalently, also as relational structures in the language L consisting of binary relations $R^1, R^2, \dots$ which denote the distances. The notions of homomorphisms, embeddings and substructures in edge-labelled graphs corresponds to the same notion of the relational structures.

Given $\mathbf{G} = (G, d) \in \mathcal{G}^\infty$ and $\mathbf{G}' = (G', d') \in \mathcal{G}^\infty$ a *homomorphism* $G \to G'$ is a function $f : G \to G'$ such that $d(x, y) = d'(f(x), f(y))$ whenever $d(x, y)$ is defined. A homomorphism f is an *embedding* (or *isometry*) if f is one-to-one and $d(x, y) = d'(f(x), f(y))$ whenever either side of the equality makes sense. A surjective embedding is an *isomorphism* and and *automorphism* is an isomorphism $\mathbf{G} \to \mathbf{G}$. A graph $\mathbf{G}$ is an *(induced) subgraph* of $\mathbf{H}$ if the identity mapping is an embedding $\mathbf{G} \to \mathbf{H}$.

2.2. Ramsey expansions

Let L^+ be a language containing the language L. By this we mean $L \subseteq L^+$ and L^+ assigns the same arity as L to all $R \in L$. Under these conditions, every structure $\mathbf{X} = (X, (R_\mathbf{X}; R \in L^+)) \in \mathrm{Rel}(L^+)$ may be viewed as a structure $\mathbf{A} = (X, (R_\mathbf{X}; R \in L)) \in \mathrm{Rel}(L)$ with some additional relations $R_\mathbf{X}$ for $R \in L^+ \setminus L$. We call $\mathbf{X}$ a *expansion* (or *lift*) of $\mathbf{A}$ and $\mathbf{A}$ is called the *reduct* (or *shadow*) of $\mathbf{X}$. In this sense the class $\mathrm{Rel}(L^+)$ is the class of all expansions of $\mathrm{Rel}(L)$, or, conversely, $\mathrm{Rel}(L)$ is the class of all shadows of $\mathrm{Rel}(L^+)$.

Definition 2.2 (Precompact expansion [21])**.** Let $\mathcal{K}^+$ be the class of lifts to L^+ of L-structures in $\mathcal{K}$. We say that $\mathcal{K}^+$ is a *precompact expansion* (or *lift*) of $\mathcal{K}$ if for every structure $\mathbf{A} \in \mathcal{K}$ there are only finitely many structures $\mathbf{A}^+ \in \mathcal{K}^+$ such that $\mathbf{A}^+$ is a lift of $\mathbf{A}$ (i.e. the shadow of $\mathbf{A}^+$ obtained by forgetting the relations in $L^+ \setminus L$ is isomorphic to $\mathbf{A}$).

Definition 2.3 (Expansion property [21])**.** Let a class $\mathcal{K}^+$ be a lift of $\mathcal{K}$. For $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ we say that $\mathcal{K}^+$ has the *expansion property* for $\mathbf{A}, \mathbf{B}$ if for every lift $\mathbf{B}^+ \in \mathcal{K}^+$ of $\mathbf{B}$ there is an embedding of every lift $\mathbf{A}^+ \in \mathcal{K}^+$ of $\mathbf{A}$ into $\mathbf{B}^+$.

$\mathcal{K}^+$ has the *expansion property* with respect to $\mathcal{K}$ if for every $\mathbf{A} \in \mathcal{K}$ there is $\mathbf{B} \in \mathcal{K}$ such that $\mathcal{K}^+$ has the expansion property for $\mathbf{A}, \mathbf{B}$.

We introduce the following stronger notion of homomorphism. An L-structure $\mathbf{A}$ is an *irreducible structure* if every pair of vertices from A is in some relation in L (the relation need not be binary: for example, a complete k-hypergraph is irreducible).

Definition 2.4 (Homomorphism-embedding [5])**.** A homomorphism $f : \mathbf{A} \to \mathbf{B}$ is a *homomorphism-embedding* if f for every irreducible substructure $\mathbf{C}$ of $\mathbf{A}$, the restriction of f to C is an embedding into $\mathbf{B}$.

While for (undirected) graphs the homomorphism and homomorphism-embedding coincide, for general relational structures they may differ.

Definition 2.5 (Completion [5])**.** Let $\mathbf{C}$ be a structure. An irreducible structure $\mathbf{C}'$ is a *(strong) completion* of $\mathbf{C}$ if there exists a one-to-one homomorphism-embedding $\mathbf{C} \to \mathbf{C}'$.

Of particular interest is the question of whether a completion of a given structure exists, such that the completed structure lies in a given class $\mathcal{K}$. If it does, we speak of its *$\mathcal{K}$-completion*.

Note that in [5] the definition of completion is weaker than the definition of strong completion. In this paper all completions will be implicitly strong.

Definition 2.6 (Locally finite subclass [5])**.** Let $\mathcal{R}$ be a class of finite irreducible structures and $\mathcal{K}$ a subclass of $\mathcal{R}$. We say that the class $\mathcal{K}$ is a *locally finite subclass of* $\mathcal{R}$ if for every $\mathbf{C}_0 \in \mathcal{R}$ there is a finite integer $n = n(\mathbf{C}_0)$ such that every structure $\mathbf{C}$ has a strong $\mathcal{K}$-completion (i.e. there exists $\mathbf{C}' \in \mathcal{K}$ that is a strong completion of $\mathbf{C}$), provided that the following conditions are satisfied:

1. there is a homomorphism-embedding from $\mathbf{C}$ to $\mathbf{C}_0$ (in other words, $\mathbf{C}_0$ is a, not necessarily strong, $\mathcal{R}$-completion of $\mathbf{C}$), and,

2. every substructure of $\mathbf{C}$ with at most n vertices has a strong $\mathcal{K}$-completion.

Theorem 2.1 (Hubička-Nešetřil [5])**.** *Let $\mathcal{R}$ be a Ramsey class of irreducible finite structures and let $\mathcal{K}$ be a hereditary locally finite subclass of $\mathcal{R}$ with strong amalgamation. Then $\mathcal{K}$ is Ramsey.*

Explicitly: For every pair of structures $\mathbf{A}, \mathbf{B}$ in $\mathcal{K}$ there exists a structure $\mathbf{C} \in \mathcal{K}$ such that

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

2.3. Coherent EPPA

Definition 2.7 (Coherent maps [33, 15])**.** Let X be a set and $\mathcal{P}$ be a family of partial bijections between subsets of X. A triple (f, g, h) from $\mathcal{P}$ is called a *coherent triple* if

$$\mathrm{Dom}(f) = \mathrm{Dom}(h), \mathrm{Range}(f) = \mathrm{Dom}(g), \mathrm{Range}(g) = \mathrm{Range}(h)$$

and

$$h = g \circ f.$$

Let X and Y be sets, and $\mathcal{P}$ and $\mathcal{Q}$ be families of partial bijections between subsets of X and between subsets of Y, respectively. A function $\varphi : \mathcal{P} \to \mathcal{Q}$ is said to be a *coherent map* if for each coherent triple (f, g, h) from $\mathcal{P}$, its image $\varphi(f), \varphi(g), \varphi(h)$ in $\mathcal{Q}$ is coherent.

Definition 2.8 (Coherent EPPA [33, 15])**.** A class $\mathcal{K}$ of finite L-structures is said to have *coherent EPPA* if $\mathcal{K}$ has EPPA and moreover the extension of partial automorphisms is coherent. That is, for every $\mathbf{A} \in \mathcal{K}$, there exists $\mathbf{B} \in \mathcal{K}$ such that $A \subseteq B$ and every partial automorphism f of $\mathbf{A}$ extends to some $\hat{f} \in \mathrm{Aut}(\mathbf{B})$ with the property that the map φ from the partial automorphisms of $\mathbf{A}$ to the automorphism of $\mathbf{B}$ given by $\varphi(f) = \hat{f}$ is coherent.

The following is a strengthening of the Herwig-Lascar Theorem [8, Theorem 2] for coherent EPPA which will be our main tool to prove EPPA in this paper.

Theorem 2.2 (Solecki-Siniora [33, 15])**.** *Let $\mathcal{O}$ be a finite family of structures, $\mathbf{A} \in \mathrm{Forb}(\mathcal{O})$, and P be a set of partial isomorphisms of A. If there exists a structure $\mathbf{M}$ containing $\mathbf{A}$ such that each element of P extends to an automorphism of $\mathbf{M}$ and moreover there is no $\mathbf{O} \in \mathcal{O}$ with homomorphism $\mathbf{O} \to \mathbf{M}$, then there exists a finite structure $\mathbf{B} \in \mathrm{Forb}(\mathcal{O})$ and $\phi : P \to \mathrm{Aut}(\mathbf{B})$ such that*

1. *$\phi(P)$ is an extension of p, and*
2. *ϕ is coherent.*

We will call a $\mathbf{B}$ with the properties as stated in Theorem 2.2 a *coherent EPPA-extension* of $\mathbf{A}$.

3. Cherlin's catalogue of metrically homogeneous graphs

Now we present Cherlin's catalogue of metrically homogeneous graphs [4] and relevant definitions. There is nothing new in this section and any divergence from [4] is a mistake.

Conjecture 3.1 (Cherlin's Metric Homogeneity Classification Conjecture [4])**.** *The countable metrically homogeneous graphs are the following.*

1. *In diameter $\delta \leq 2$: the homogeneous graphs, classified by Lachlan and Woodrow [2].*
2. *In diameter $\delta \geq 3$:*
   (a) *The finite ones, classified by Cameron [34].*
   (b) *Macpherson's regular tree-like graphs $T_{m,n}$ with $m, n \leq \infty$, $m, n \geq 2$,*
   (c) *The Fraïssé limits of amalgamation classes of the form $\mathcal{A}_3 \cap \mathcal{A}_H$ with $\mathcal{A}_3$ 3-constrained and $\mathcal{A}_H$ of Henson type or antipodal Henson type.*

3.1. *3-constrained spaces $\mathcal{A}_3$*

Recall the numerical parameters δ, K_1, K_2, C_0 and C_1 introduced in Definition 1.6 to describe the 3-constrained spaces.

Different numerical parameters can be used to describe the same classes of structures. To avoid this redundancy we will assume the following constraints which describe meaningful sets of parameters.

Definition 3.1 (Acceptable numerical parameters)**.** A sequence of parameters $(\delta, K_1, K_2, C_0, C_1)$ is *acceptable* if it satisfies the following conditions:

- $3 \leq \delta \leq \infty$;
- $1 \leq K_1 \leq K_2 \leq 2\delta$ or $K_1 = \infty$ and $K_2 = 0$;
- $2\delta < C_0, C_1$ and $C_0, C_1 \leq 3\delta + 2$. Here C_0 is even and C_1 is odd;
- If $K_1 = \infty$ (the bipartite case) then $C_1 = 2\delta + 1$.

Theorem 3.2 (Cherlin's Admissibility Theorem [4])**.** *Let $(\delta, K_1, K_2, C_0, C_1)$ be an acceptable sequence of parameters (in particular, $\delta \geq 3$). Then the associated class $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ is an amalgamation class if and only if one of the following three groups of conditions is satisfied, where we write C for $\min(C_0, C_1)$ and C' for $\max(C_0, C_1)$:*

(I) *$K_1 = \infty$ (the bipartite case; so $K_2 = 0$ and $C_1 = 2\delta + 1$).*

(II) *$K_1 < \infty, C \leq 2\delta + K_1$, and*

- *$C = 2K_1 + 2K_2 + 1$;*
- *$K_1 + K_2 \geq \delta$;*
- *$K_1 + 2K_2 \leq 2\delta - 1$, and:*

K_1	K_2	C_0	C_1	M	Case	$\mathcal{S}$	Structure
∞	0	8	7	–	(I)	$\emptyset$	Bipartite antipodal
∞	0	10	7	2	(I)	$\emptyset$	Bipartite
1	2	8	7	–	(IIA)	$\emptyset$	Antipodal
1	2	10	9	2	(III)	K_n	No $\delta\delta\delta$, $1\delta\delta$ triangles
1	2	10	11	2	(III)	K_n and/or I_m	No $1\delta\delta$ triangles
1	3	8	9	2	(III)	$\emptyset$	No 5-anticycle
1	3	10	9	2	(III)	Any w/o I_3	No $\delta\delta\delta$ triangles
1	3	10	11	$2,3$	(III)	Any	All metric spaces
2	2	10	9	2	(III)	$\emptyset$	No $\delta\delta\delta$, $1\delta\delta$, 111
2	2	10	11	2	(III)	I_m	No $1\delta\delta$, 111 triangles
2	3	10	9	2	(III)	Any w/o K_3,I_3	No $\delta\delta\delta$, 111 triangles
2	3	10	11	$2,3$	(III)	Any w/o K_3	No 111 triangles
3	3	10	11	3	(III)	$\emptyset$	No 5-cycle

Table 1: All admissible parameters for $\delta = 3$ with the set $\mathcal{S}$ of Henson constraints limited to meaningful choices [35, Table 2] and parameter M satisfying Definitions 4.4 and 6.1. K_n denotes the n-clique (i.e. the metric space with n vertices and all distances 1) and I_n is the n-anticlique (all distances δ). The second column lists the possible choices for magic distances (see Definition 6.1)

(IIA) $C' = C + 1$, *or*

(IIB) $C' > C + 1, K_1 = K_2$, *and* $3K_2 = 2\delta - 1$.

(III) $K_1 < \infty$, $C > 2\delta + K_1$, *and:*

- $K_1 + 2K_2 \geq 2\delta - 1$ *and* $3K_2 \geq 2\delta$*;*
- *If* $K_1 + 2K_2 = 2\delta - 1$ *then* $C \geq 2\delta + K_1 + 2$*;*
- *If* $C' > C + 1$ *then* $C \geq 2\delta + K_2$.

A sequence of parameters $(\delta, K_1, K_2, C_0, C_1)$ is called *admissible* if and only if it satisfies one of the three sets of conditions in Theorem 3.2.

Example. All admissible parameters with $\delta = 3$ are listed in Table 1.

3.2. *Henson constraints*

Suppose $\delta \geq 3$. A $(1, \delta)$-*space* is a metric space in which all distances are 1 or δ; thus the relation $d(x, y) \leq 1$ is an equivalence relation, and the classes lie at mutual distance δ. A $(1, \delta)$-space will also be called a *Henson constraint*.

A *clique* is a metric space $\mathbf{K} = (K, d)$ where $d(u, v) = 1$ for every $u \neq v$. $\mathbf{K}^* = (M, d)$ is an *antipodal companion of the clique* $\mathbf{K}$ if there exists $S \subset M$ such that for distinct vertices u, v, $d(u, v) = \delta - 1$ if $u \in S, v \notin S$ or vice versa and $d(u, v) = 1$ otherwise.

Definition 3.2 (Acceptable parameters with Henson constraints)**.** The sequence of parameters $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ is *acceptable* if

1. $(\delta, K_1, K_2, C_0, C_1)$ is an acceptable sequence of numerical parameters.

2. $\mathcal{S}$ is a set of (δ-)Henson constraints, i.e., a set of $(1, \delta)$-spaces if $C_1 > 2\delta + 1$ or $C_0 > 2\delta + 2$, and a set of cliques and their antipodal companions if $C_1 = 2\delta + 1$ and $C_0 = 2\delta + 2$ (cf. Definition 7.2).

3. $\mathcal{S}$ is *irredundant* in the sense that no constraint in $\mathcal{S}$ contains triangles forbidden in $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$, and every constraint in $\mathcal{S}$ consists of at least 4 vertices.

Definition 3.3 (Admissible parameters with Henson constraints)**.** Let $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ be an acceptable sequence of parameters, where $\mathcal{S}$ is a set of δ-Henson constraints. Let $C = \min(C_0, C_1)$ and $C' = \max(C_0, C_1)$. The sequence is *admissible* if

1. The sequence $(\delta, K_1, K_2, C_0, C_1)$ is admissible.

2. If $C = 2\delta + 1$, $K_1 < \infty$, and $\mathcal{S}$ is nonempty, then $\delta \geq 4$ and $\mathcal{S}$ consists of a clique and all its antipodal companions.

3. If $K_1 \leq \infty$ and $C > 2\delta + K_1$ (Case (III)), then

   - If $K_1 = \delta$ then $\mathcal{S}$ is empty;
   - If $C = 2\delta + 2$ then $\mathcal{S}$ is empty.

4. Primitive 3-constrained spaces

In this section we will work with fixed admissible parameters $\delta < \infty$, K_1, K_2, C_0 and C_1, such that the associated homogeneous metric space $\Gamma^{\delta}_{K_1,K_2,C_0,C_1}$ is primitive, i.e. it has no non-trivial definable equivalence relation on it. As one can verify, those are exactly the parameters where the class $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ has strong amalgamation and contains at least one triangle with odd perimeter. Recall that we denote $C = \min(C_0, C_1)$, and $C' = \max(C_0, C_1)$. Then we can characterize the parameters as follows:

Definition 4.1. The admissible parameters $\delta < \infty$, K_1, K_2, C_0 and C_1 are *primitive* when they satisfy case (II) or (III) of Theorem 3.2 and moreover do not form an antipodal space.

In other words, the triangle $\delta\delta a$ for some $a > 1$ is permitted and thus

$$C_0, C_1 \geq 2\delta + 2.$$

For many of the lemmas in this section, the inequalities $C_0, C_1 \geq 2\delta + 2$ will be an essential assumption.

We would like to recall that $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ was defined in Definition 1.6 as the class of all metric spaces that satisfy some constraints on its triangles, i.e. its 3-element subspaces. In the following it will often be more convenient to think of $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ as the class of metric spaces that do not embed any triangle violating those constraints — we are going to refer to such triangles as *forbidden triangles*. We also will slightly abuse notation and use the term triangle for both triples of vertices u, v, w and for triples of edges a, b, c with $a = d(u, v)$, $b = d(v, w)$, $c = d(u, w)$. By Definition 1.6 a triangle abc is forbidden if it satisfies one of the following conditions:

Non-metric: a, b, c is forbidden if $a + b < c$,

K_1-bound: $a + b + c < 2K_1 + 1$ and $a + b + c$ is odd,

K_2-bound: $b + c \geq 2K_2 + a$ and $a + b + c$ is odd and $a \leq b, c$,

C_1-bound: $a + b + c \geq C_1$ and $a + b + c$ is odd,

C_0-bound: $a + b + c \geq C_0$ and $a + b + c$ is even,

C-bound: If $|C_0 - C_1| = 1$, the C_1-bound and C_0-bound can be expressed together as $a + b + c \geq C$, where $C = \min(C_0, C_1)$.

Triangles that are not forbidden will be called *allowed*.

4.1. Generalised completion algorithm for 3-constrained classes

Put $\mathcal{D} = \{1, 2, \ldots \delta\}^2$ be the collection of (ordered) pairs. It is more natural to consider unordered pairs, but notationally easier to consider ordered pairs. We will refer to elements of $\mathcal{D}$ as *forks*.

Consider a δ-bounded variant of shortest path completion, where we assume that the input graphs contain no distances greater than δ and in the output all edges longer than δ are replaced by an edge of that length. There is an alternative formulation of this completion: For a fork $\vec{f} = (a, b)$, define $d^{+}(\vec{f}) = \min(a + b, \delta)$. In the i-th step look at all incomplete forks $\vec{f}$ (i.e. triples of vertices u, v, w such that exactly two edges are present) such that $d^{+}(\vec{f}) = i$ and define the length of the missing edge to be i.

This algorithm proceeds by first adding edges of length 2, then edges of length 3 and so on up to edges of length δ and has the property that out of all metric completions of a given graph, every edge of the completion yielded by this algorithm is as close to δ as possible.

It makes sense to ask what happens if, instead of trying to make each edge as close to δ as possible, one would try to make each edge as close to some parameter M as possible. For M in a certain range, such an algorithm exists. For each fork $\vec{f} = (a, b)$ one can define $d^+(\vec{f}) = a + b$ and $d^-(\vec{f}) = |a - b|$, i.e. the largest and the smallest possible distance that can metrically complete the fork $\vec{f}$. The generalised algorithm will complete $\vec{f}$ by $d^+(\vec{f})$ if $d^+(\vec{f}) < M$, by $d^-(\vec{f})$ if $d^-(\vec{f}) > M$ and by M otherwise. It turns out that there is a good permutation π of $\{1, \ldots, \delta\}$, such that if one adds the distances in the order prescribed by the permutation, this generalised algorithm will produce a correct completion whenever one exists. It is easy to check that the choice $M = \delta$ and $\pi = \mathrm{id}_\delta$ corresponds to the shortest path completion algorithm.

Definition 4.2 (Completion algorithm)**.** Given $c \geq 1$, $\mathcal{F} \subseteq \mathcal{D}$, and a graph $\mathbf{G} = (G, d) \in \mathcal{G}^\delta$, we say that $\mathbf{G}' = (G, d')$ is the $(\mathcal{F}, c)$*-completion* of $\mathbf{G}$ if $d'(u, v) = d(u, v)$ whenever u, v is an edge of $\mathbf{G}$ and $d'(u, v) = c$ if u, v is not an edge of $\mathbf{G}$ and there exist $(a, b) \in \mathcal{F}$, $w \in G$ such that $\{d(u, w), d(v, w)\} = \{a, b\}$. There are no other edges in $\mathbf{G}'$.

Given $1 \leq M \leq \delta$, a one-to-one function $t : \{1, 2, \ldots, \delta\} \setminus \{M\} \to \mathbb{N}$ and a function $\mathbb{F}$ from $\{1, 2, \ldots, \delta\} \setminus \{M\}$ to the power set of $\mathcal{D}$, we define the $(\mathbb{F}, t, M)$*-completion* of $\mathbf{G}$ as the limit of a sequence of edge-lablelled graphs $\mathbf{G}_1, \mathbf{G}_2, \ldots$ such that $\mathbf{G}_1 = \mathbf{G}$ and $\mathbf{G}_{k+1} = \mathbf{G}_k$ if $t^{-1}(k)$ is undefined and $\mathbf{G}_{k+1}$ is the $(\mathbb{F}(t^{-1}(k)), t^{-1}(k))$-completion of $\mathbf{G}_k$ otherwise, with every pair of vertices not forming an edge in this limit set to distance M.

We will call the vertex w from Definition 4.2 the *witness of the edge* u, v. The function t is called the *time function* of the completion because edges of length a are inserted to $\mathbf{G}_{t(a)}$ the $t(a)$-th step of the completion. If for a $(\mathbb{F}, t, M)$-completion and distances a, c there is a distance b such that $(a, b) \in \mathbb{F}(c)$ (i.e. the algorithm might complete a fork (a, b) with distance c), we say that *c expands to a*.

Definition 4.3 (Magic distances)**.** Let $M \in \{1, 2, \ldots, \delta\}$ be a distance. We say that M is *magic* (with respect to $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$) if

$$\max\left(K_1, \left\lceil \frac{\delta}{2} \right\rceil\right) \leq M \leq \min\left(K_2, \left\lfloor \frac{C - \delta - 1}{2} \right\rfloor\right).$$

Note that for primitive admissible parameters $(\delta, K_1, K_2, C_0, C_1)$ such an M always exists.

Observation 4.1. *The set of magic distances (with respect to $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$) is*

$$S = \{1 \leq a \leq \delta : aab \text{ is allowed for all } 1 \leq b \leq \delta\}.$$

Proof. If a distance a is in S, then $a \geq K_1$ (otherwise the triangle $aa1$ has perimeter $2a + 1$, which is odd and smaller than $2K_1 + 1$, hence forbidden by the K_1 bound), $a \geq \left\lceil \frac{\delta}{2} \right\rceil$ (otherwise the triangle $aa\delta$ is non-metric), $a \leq \left\lfloor \frac{C-\delta-1}{2} \right\rfloor$ (otherwise the triangle aab has perimeter C for $b = C - 2a \leq \delta$), and $a \leq K_2$ (otherwise the triangle $aa1$ has odd perimeter and $2a \geq 2K_2 + 1$, hence is forbidden by the K_2 bound). The other implication follows from the definition of $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$. □

Let M be a magic distance and $x \in \{1, \ldots, \delta\} \setminus \{M\}$. Define

$$\mathcal{F}^+_x = \{(a, b) \in \mathcal{D} : a + b = x\},$$

$$\mathcal{F}^-_x = \{(a, b) \in \mathcal{D} : |a - b| = x\},$$

$$\mathcal{F}^C_x = \{(a, b) \in \mathcal{D} : C - 1 - a - b = x\}.$$

We further denote

$$\mathbb{F}_M(x) = \begin{cases} \mathcal{F}^+_x \cup \mathcal{F}^C_x & \text{if } x < M \\ \mathcal{F}^-_x & \text{if } x > M. \end{cases}$$

For a magic distance M, we also define the function $t_M : \{1, \ldots, \delta\} \setminus \{M\} \to \mathbb{N}$ as

$$t_M(x) = \begin{cases} 2x - 1 & \text{if } x < M \\ 2(\delta - x) & \text{if } x > M. \end{cases}$$

Forks and how they are completed according to $\mathbb{F}_M$ are schematically depicted in Figure 2.

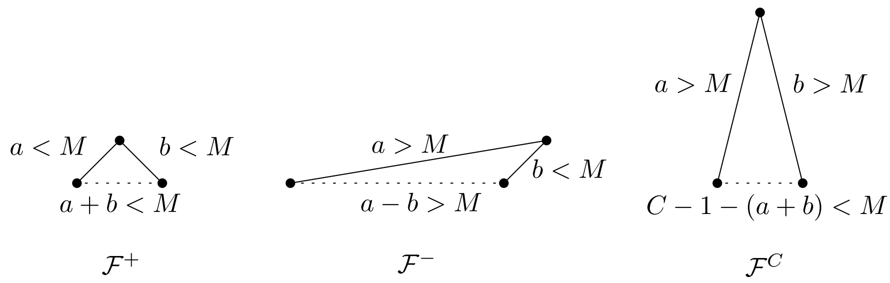

Figure 2: Forks used by $\mathbb{F}_M$.

Definition 4.4 (Completion with magic parameter M)**.** Let M be a magic distance satisfying the following extra conditions:

1. If the parameters satisfy Case (III) with $K_1 + 2K_2 = 2\delta - 1$, then $M > K_1$;
2. if the parameters satisfy Case (III) and further $C' > C + 1$ and $C = 2\delta + K_2$, then $M < K_2$.

We then call the $(\mathbb{F}_M, t_M, M)$-completion (of $\mathbf{G}$) the *completion (of* $\mathbf{G}$*) with magic parameter* M.

Our main goal of the following section is the proof of Theorem 4.9 that shows that the completion of $\mathbf{G}$ with magic parameter M lies in $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ if and only if $\mathbf{G}$ has some completion in $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$.

The two extra conditions in Definition 4.4 are a way to deal with certain extremal choices of admissible primitive parameters $(\delta, K_1, K_2, C_0, C_1)$. We are going to check that also with those extra conditions, there will always exist a suitable magic distance:

Lemma 4.2. *For primitive parameters* $(\delta, K_1, K_2, C_0, C_1)$ *there is always an* M *satisfying Definitions 4.3 and 4.4.*

Proof. For Case (III) with $K_1 + 2K_2 = 2\delta - 1$, we proceed as follows: From admissibility, we have

$$K_1 + 2K_2 = 2\delta - 1$$
$$3K_2 \geq 2\delta$$

so we conclude that $K_1 < K_2$. From this information and $K_1 + 2K_2 = 2\delta - 1$, we derive $K_1 < \frac{2}{3}\delta$. We know that $\delta - 1 \geq \frac{2}{3}\delta$ for $\delta \geq 3$ and $K_1 \leq \delta - 2$, so $\left\lfloor \frac{C-\delta-1}{2} \right\rfloor \geq \left\lfloor \frac{\delta+K_1+1}{2} \right\rfloor \geq \frac{\delta+K_1}{2}$. Hence, $K_1 < \left\lfloor \frac{C-\delta-1}{2} \right\rfloor$ and there is always a magic number greater than K_1.

In Case (III) with $C' > C + 1$ and $C = 2\delta + K_2$, we know from admissibility that $C > 2\delta + K_1$, so $K_2 > K_1$. Now we need $\left\lceil \frac{\delta}{2} \right\rceil < K_2$. For $\delta \geq 3$, the inequality $\left\lceil \frac{\delta}{2} \right\rceil \leq \frac{2}{3}\delta$ holds with equality only for $\delta = 3$. Admissibility tells us $3K_2 \geq 2\delta$. Now, if $\delta > 3$ or $K_2 \neq \frac{2}{3}\delta$, it follows that $\left\lceil \frac{\delta}{2} \right\rceil < K_2$.

The only remaining possibility is $\delta = 3$ and $K_2 = 2$, which implies $C = 8$ and $K_1 = 1$, which gives us $2K_2 + K_1 = 5 = 2\delta - 1$. The admissibility condition $C \geq 2\delta + K_1 + 2$ then yields $C \geq 9$, a contradiction. Hence there always is a magic number smaller than K_2.

If both these situations occur simultaneously, then we further require M with $K_1 < M < K_2$. But that follows as $C = 2\delta + K_2$ and whenever $K_1 + 2K_2 = 2\delta - 1$, from admissibility we have $C \geq 2\delta + K_1 + 2$, hence $K_2 \geq K_1 + 2$. □

Observe that the algorithm only makes use of C, δ and M. The interplay of individual parameters of algorithm is schematically depicted in Figure 3.

Example (Case (IIB))**.** In our proofs, Case (IIB) will often form a special case. The smallest (in terms of diameter) set of acceptable parameters that is in Case (IIB) is:

$$\delta = 5, C = C_1 = 13, C' = C_0 = 16, K_1 = K_2 = \frac{2\delta - 1}{3} = 3.$$

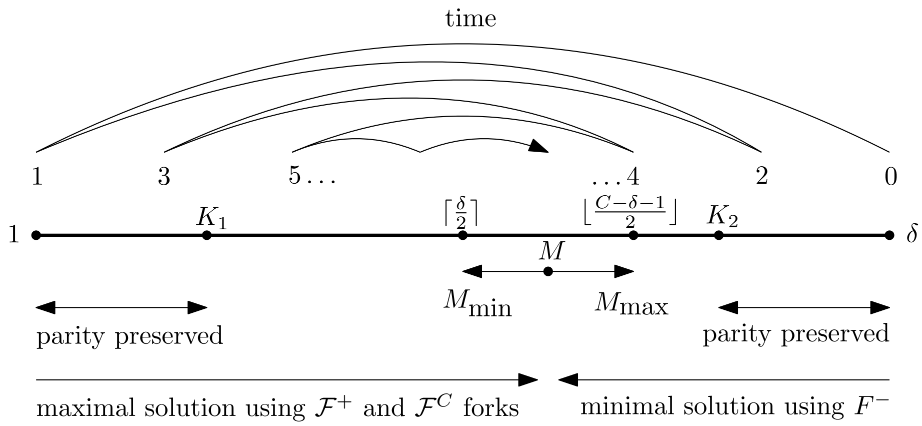

Figure 3: A sketch of the main parameters of the completion algorithm, the Optimality Lemma 4.6 and the Parity Lemma 4.7.

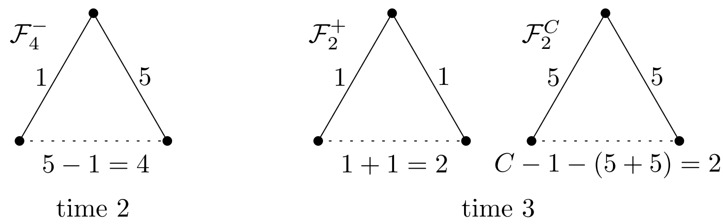

Figure 4: Forks considered by the algorithm to complete to $\mathcal{A}^5_{3,3,16,13}$ with $M = 3$.

Here $M = 3$, and it is the only choice for a magic number.

Forbidden triangles are those that are non-metric (113, 114, 115, 124, 125, 135, 225), or rejected for the K_1-bound (111, 122), the K_2-bound (144, 155, 245), or the C_1-bound (355, 445, 555). There are no triangles forbidden by C_0. Table 2 lists all possible completions of forks, with the completion preferred by our algorithm in bold type. Completions for forks in this class depicted in Figure 4. Notably, the magic number $M = 3$ is chosen for all forks except $(1, 1)$, which is completed by $d^+((1, 1)) = 2$, $(1, 5)$, which is completed by $d^-((1, 5)) = 4$, and $(5, 5)$, which is a C-bound case. Those cases are the only forks where $M = 3$ cannot be chosen, so instead the algorithm chooses the nearest possible completion. What makes Case (IIB) special is the situation where one can choose $M - 1$ or $M + 1$ but not M when completing a fork (for $\delta = 5$ it is the fork $(5, 5)$).

The algorithm will thus effectively run in three steps. First (at time 2) it will complete all forks $(1, 5)$ with distance 4, next (at time 3) it will complete all forks $(1, 1)$ and $(5, 5)$ with distance 2 and finally it will turn all non-edges into edges of distance 3. Examples of runs of this algorithm are given later, see Figures 6 and 7.

4.2. *What do forbidden triangles look like?*

The majority of the proofs in the following sections assume that the completion algorithm with magic parameter M introduces some forbidden triangle and then argue that the triangle must be forbidden in any completion, hence the input structure has no completion into $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$. In such an argument it will be helpful to know how the different types of forbidden triangles relate to the magic parameter M. Therefore, in the following paragraphs we will study how triangles forbidden by different bounds are related to the magic parameter M. We will use a, b, c for the lengths of the sides of the triangle and without loss of generality assume $a \le b \le c$. All conclusions are summarised in Figure 5.

non-metric: If $a + b < c$, then $a < M$, because otherwise $a + b \ge 2M \ge \delta$.

K_1-bound: If $a + b + c < 2K_1 + 1$, $a + b + c$ is odd and abc is metric, then $a, b, c < K_1 \le M$, because if $c \ge K_1$, then from the metric condition $a + b \ge c \ge K_1$ and hence $a + b + c \ge 2K_1$, for odd $a + b + c$ this means $a + b + c \ge 2K_1 + 1$.

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	**2**	1, **3**	2, **3**, 4	**3**, 5	**4**
$i=2$		2, **3**, 4	1, 2, **3**, 4, 5	2, **3**, 4	**3**, 5
$i=3$			1, 2, **3**, 4, 5	1, 2, **3**, 4, 5	2, **3**, 4
$i=4$				2, **3**, 4	1, **3**, 5
$i=5$					**2**, 4

Table 2: Possible ways to complete (i, j) forks, the bold number is the completion with magic parameter $M = 3$.

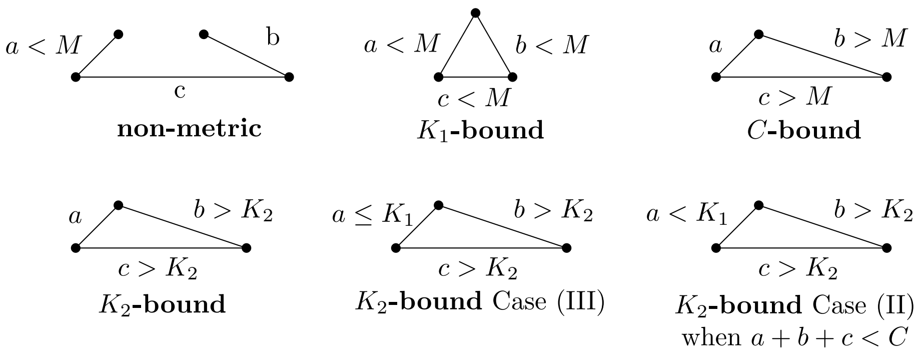

Figure 5: Types of forbidden triangles.

C-bound: If $a + b + c \geq C$ then $b, c > M$. Suppose for a contradiction that $a, b \leq M$. We then have $a + b \geq C - c \geq C - \delta$, but on the other hand $a + b \leq 2M \leq 2\left\lfloor \frac{C-\delta-1}{2} \right\rfloor \leq C - \delta - 1$, which together yield $C - \delta - 1 \geq C - \delta$, a contradiction. Note that in some cases $C' \neq C + 1$, but this observation still holds, as it only uses $a + b + c \geq C$.

K_2-bound: If abc is a metric triangle with odd perimeter, then abc breaks the K_2 condition if and only if $b + c \geq 2K_2 + a + 1$ (the 1 on the right side comes from $a + b + c$ being odd and all distances being integers). Then $b, c > K_2$, because if $b \leq K_2$, from metricity we have $c \leq a + b$, hence $a + 2K_2 \geq (a + b) + b \geq c + b \geq 2K_2 + a + 1$, a contradiction.

Moreover, in Case (III) of Theorem 3.2 we have $a \leq K_1$ because if $a > K_1$, we have $b + c \geq 2K_2 + a + 1 > 2K_2 + K_1 + 1$ and from admissibility conditions for Case (III) we have $2K_2 + K_1 \geq 2\delta - 1$, which gives $b + c > 2\delta$, a contradiction. (Note that if $2K_2 + K_1 > 2\delta - 1$, we have $a < K_1$.)

Finally if $a + b + c < C$ (which is stronger than not being forbidden by the C bound, as it also includes the $C' > C + 1$ cases) and we are in Case (II) (where $C = 2K_1 + 2K_2 + 1$), we get $a < K_1$, because if $a \geq K_1$, we would get $a + b + c \geq 2K_2 + 2a + 1 \geq 2K_2 + 2K_1 + 1 = C$.

4.3. Basic properties of the completion algorithm

In this section we develop several technical observations about the algorithm which will be used in Section 4.4 to show the main result about correctness of the algorithm.

Recall the definition of t_M and $\mathbb{F}_M$:

$$t_M(x) = \begin{cases} 2x - 1 & \text{if } x < M \\ 2(\delta - x) & \text{if } x > M. \end{cases}$$

$$\mathbb{F}_M(x) = \begin{cases} \mathcal{F}_x^+ \cup \mathcal{F}_x^C & \text{if } x < M \\ \mathcal{F}_x^- & \text{if } x > M. \end{cases}$$

Intuitively, the function $\mathbb{F}_M$ selects the forks that will be completed to triangles with an edge of type $t_M^{-1}(x)$ at time x. At time 0 it looks for forks that can be completed with distance δ, then with distance 1, jumping back and forth on the distance set and approaching M (cf. Figure 3). Later, we will see that all forks that cannot be completed with M are in some $\mathbb{F}_M(x)$.

Now we shall precisely state and prove that t_M gives a suitable injection for the algorithm, as claimed before Definition 4.2.

Lemma 4.3 (Time Consistency Lemma)**.** *Let a, b be distances different from M. If a expands to b, then $t_M(a) > t_M(b)$.*

Proof. We consider three types of forks used by the algorithm:

$\mathcal{F}^+$: If $a < M$ and $\mathcal{F}_a^+ \neq \emptyset$, then $b < a < M$, hence $t_M(b) < t_M(a)$.

$\mathcal{F}^C$: If $a < M$ and $\mathcal{F}_a^C \neq \emptyset$, then we must have $b, c > M$. Otherwise, if for instance $b \leq M$, then $C - \delta - 1 \leq C - 1 - c = a + b < 2M \leq 2\left\lfloor \frac{C-\delta-1}{2} \right\rfloor$, a contradiction. As $C \geq 2\delta + 2$, we obtain the inequality $b = (C-1) - c - a \geq (2\delta + 1) - \delta - a = \delta + 1 - a$. Hence $t_M(b) \leq 2(a-1) < 2a - 1 = t_M(a)$.

$\mathcal{F}^-$: Finally, we consider the case where $a > M$ and $\mathcal{F}_a^- \neq \emptyset$. Then either $a = b - c$, which implies $b > a > M$ and thus $t_M(b) < t_M(a)$, or $a = c - b$, which means $b = c - a \leq \delta - a$. Because of $a > M \geq \left\lceil \frac{\delta}{2} \right\rceil$, we have $b < M$. So $t_M(b) \leq 2(\delta - a) - 1 < 2(\delta - a) = t_M(a)$.

□

The forks in the range of $\mathbb{F}_M$ include by definition all forks that cannot be completed with M.

Lemma 4.4 ($\mathbb{F}_M$ Completeness Lemma)**.** *Let $\mathbf{G} \in \mathcal{G}^\delta$ and $\overline{\mathbf{G}}$ be its completion with a magic parameter M. If there is a forbidden triangle (w.r.t. $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$) or a triangle with perimeter at least C in $\overline{\mathbf{G}}$ with an edge of length M, then this edge is also present in $\mathbf{G}$.*

Observe that for $C' \neq C + 1$, this lemma is talking not only about forbidden triangles, but about all triangles with perimeter at least C.

Proof. By Observation 4.1 no triangle of type aMM is forbidden, so suppose that there is a forbidden triangle abM in $\overline{\mathbf{G}}$ such that the edge of length M is not in $\mathbf{G}$. For convenience define $t_M(M) = \infty$, which corresponds to the fact that edges of length M are added in the last step.

non-metric: If abM is non-metric then either $a + b < M$ or $|a - b| > M$. By Lemma 4.3 we have in both cases that $t_M(a + b)$ (respectively, $t_M(|a - b|)$) is greater than both $t_M(a)$ and $t_M(b)$. Therefore the completion algorithm would chose $a + b$ (resp, $|a - b|$) as the length of the edge instead of M.

K_1-bound: Now that we know that abM is metric, we also know that it is not forbidden by the K_1 bound, because $M \geq K_1$.

C-bound: If $a + b + M \geq C$ (which includes all the triangles forbidden by C_0 or C_1 bounds), then $t_M(C - 1 - a - b) > t_M(a), t_M(b)$ by Lemma 4.3, so the algorithm would set $C - 1 - a - b$ instead of M as the length of the third edge.

K_2-bound: Finally we deal with the K_2 bound. Suppose that abM is metric, its perimeter is less than C, and it is forbidden by the K_2 bound. From Section 4.2 we have that the two long edges have to be longer that K_2, and the shortest edge is at most K_1 with equality only in Case (III) with $K_1 + 2K_2 = 2\delta - 1$.

As $M \leq K_2$, we know that M is the shortest edge. But also $M \geq K_1$, hence this situation can happen only when K_1 is the length of the shortest edge, which is only in Case (III) with $K_1 + 2K_2 = 2\delta - 1$. But from Definition 4.4 we have in this case $M > K_1$. Hence this situation never occurs.

□

It may seem strange that the algorithm does not differentiate between C_0 and C_1. The following observation justifies this by showing that in the case where $C' > C + 1$, these bounds have a relatively limited effect on the run of the algorithm.

Observation 4.5. *If $C' > C + 1$, then either $\mathcal{F}^C_x$ is empty for all $x < M$ or the parameters satisfy (IIB). In the latter case, only $\mathcal{F}^C_{M-1} = \{(\delta, \delta)\}$ is non-empty. Furthermore, in this case $t_M(M - 1)$ is the maximum of the time-function. This implies that (δ, δ)-forks are completed to $M - 1$ in the penultimate step of the completion algorithm.*

"penultimate", how fancy! -Mike

Proof. Consider a fork $(a, b) \in \mathcal{F}^C_x$ and the cases where $C' > C + 1$ is allowed.

In Case (III) with $C' > C + 1$ we have (by admissibility) $C \geq 2\delta + K_2$, so $x = C - 1 - a - b \geq K_2 - 1$ with equality only for $C = 2\delta + K_2$. From the extra condition for a magic parameter in Definition 4.4, we get that $M < K_2$.

In Case (IIB) we have $M = K_2 = K_1 = \frac{2\delta-1}{3}$, hence $C = 2K_1 + 2K_2 + 1 = 2\delta + K_2$, thus again we have $C - 1 - a - b \geq K_2 - 1$. This means that the only fork in $\mathcal{F}^C_{M-1}$ is going to be (δ, δ), which will be completed by $K_2 - 1 = M - 1$.

In order to see that $t_M(M - 1)$ is maximal, let a be a distance different from M and $M - 1$. We have $3M = 3K_2 = 2\delta - 1$, so by definition $t_M(M - 1) = 2M - 3$ and $t_M(M + 1) = 2\delta - 2M - 2$, so we want $2M - 3 > 2\delta - 2M - 2$, or $4M > 2\delta + 1$ which is true for $\delta \geq 5$ and this always holds in Case (IIB). So $t_M(M - 1) > t_M(M + 1)$ and therefore $t_M(M - 1) > t_M(a)$. □

Lemma 4.6 (Optimality Lemma). *Let $\mathbf{G} = (G, d) \in \mathcal{G}^\delta$ such that it has a completion in $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$. Denote by $\overline{\mathbf{G}} = (G, \bar{d})$ the completion of $\mathbf{G}$ with a magic parameter M and let $\mathbf{G}' = (G, d') \in \mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ be an arbitrary completion of $\mathbf{G}$. Then for every pair of vertices $u, v \in G$ one of the following holds:*

1. *$d'(u, v) \geq \bar{d}(u, v) \geq M$,*
2. *$d'(u, v) \leq \bar{d}(u, v) \leq M$,*
3. *the parameters $(\delta, K_1, K_2, C_0, C_1)$ satisfy Case (IIB), $\bar{d}(u, v) = M - 1$, $d'(u, v) > M$ and $d'(u, v)$ has the same parity as $\bar{d}(u, v)$.*

Note that for $\bar{d}(u, v) = M$ the statement trivially holds.

Proof. Suppose that the statement is not true and take any witness $\mathbf{G}' = (G, d')$ (i.e. a completion of $\mathbf{G}$ into $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ such that there is a pair of vertices violating the statement). Recall that the completion with a magic parameter M is defined as a limit of a sequence $\mathbf{G}_1, \mathbf{G}_2, \ldots$ of edge-labelled graphs such that $\mathbf{G}_1 = \mathbf{G}$ and each two subsequent graphs differ at most by adding edges of a single distance.

Take the smallest i such that in the graph $\mathbf{G}_i = (G, d_i)$ there are vertices $u, v \in G$ with $d_i(u, v) > M$ and $d_i(u, v) > d'(u, v)$ or $d_i(u, v) < M$ and $d_i(u, v) < d'(u, v)$. Let $w \in G$ be the witness of $d_i(u, v)$. In Case (IIB), by Observation 4.5 edges of length $M - 1$ are added in the last step of our completion algorithm. Therefore we know that the distances $d_{i-1}(u, w)$ and $d_{i-1}(v, w)$ satisfy the optimality conditions in point 1 or 2.

We shall distinguish three cases, based on whether $d_i(u, v)$ was introduced by $\mathcal{F}^-$, $\mathcal{F}^+$ or $\mathcal{F}^C$:

$\mathcal{F}^-$ case. We have $M < d_i(u, v) = |d_{i-1}(u, w) - d_{i-1}(v, w)|$. Without loss of generality let us assume $d_{i-1}(u, w) > d_{i-1}(v, w)$, which means that $d_{i-1}(u, w) > M$ and $d_{i-1}(v, w) < M$ (as $M \geq \left\lceil \frac{\delta}{2} \right\rceil$). From the minimality of i, it follows that $d'(u, w) \geq d_{i-1}(u, w)$ and $d'(v, w) \leq d_{i-1}(v, w)$. Since $\mathbf{G}'$ is metric we have $d_i(u, v) = d_{i-1}(u, w) - d_{i-1}(v, w) \leq d'(u, w) - d'(v, w) \leq d'(u, v)$, which is a contradiction.

$\mathcal{F}^+$ case. We have $M > d_i(u, v) = d_{i-1}(u, w) + d_{i-1}(v, w)$, hence $d_{i-1}(u, w), d_{i-1}(v, w) < M$. By the minimality of i we have $d'(u, w) \leq d_{i-1}(u, w)$ and $d'(v, w) \leq d_{i-1}(v, w)$. Since $\mathbf{G}'$ is metric, we get $d'(u, v) \leq d_i(u, v)$, which contradicts our assumptions.

$\mathcal{F}^C$ case. We have $M > d_i(u, v) = C - 1 - d_{i-1}(u, w) - d_{i-1}(v, w)$.

First suppose that $C' = C + 1$. Recall that, by the admissibility of C, we have $C - 1 \geq 2\delta + 1$ and $M \leq \left\lfloor \frac{C-\delta-1}{2} \right\rfloor$. Thus we get $d_{i-1}(u, w), d_{i-1}(v, w) > M$ (otherwise, if, say, $d_{i-1}(u, w) \leq M$, we obtain the contradiction $C - \delta - 1 \geq 2M > d_{i-1}(u, w) + d_i(u, v) = C - 1 - d_{i-1}(v, w) \geq C - \delta - 1$). So again $d'(u, w) \geq d_{i-1}(u, w)$ and $d'(v, w) \geq d_{i-1}(v, w)$, which means that the triangle u, v, w in $\mathbf{G}'$ is forbidden by the C bound, which is absurd as $\mathbf{G}'$ is a completion of $\mathbf{G}$ in $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$.

It remains to discuss the case where $C' > C + 1$. By Observation 4.5, we only need to consider Case (IIB), $d_i(u, v) = K_2 - 1 = M - 1$ and $d_{i-1}(u, w) = d_{i-1}(v, w) = \delta$. By our assumption we have $d'(u, v) > d_i(u, v)$. Hence if $d'(u, v) \geq M$ it has to have the same parity as $d_i(u, v)$ (otherwise the triangle u, v, w would be forbidden in $\mathbf{G}'$ by the C bound). □

Next we show that the algorithm initially runs in a way that preserves the parity of completions to $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$.

Lemma 4.7 (Parity Lemma). *Let $\mathbf{G}$, $\overline{\mathbf{G}}$ and $\mathbf{G}'$ be as in Lemma 4.6. Then for every pair of vertices $u, v \in G$ such that either $\bar{d}(u, v) \leq \min(K_1, M - 1)$ or $\bar{d}(u, v) \geq \max(K_2, M + 1)$, at least one of the following holds:*

1. *The parity of $\bar{d}(u, v)$ is the same as the parity of $d'(u, v)$;*
2. *the parameters come from Case (III), $C = 2\delta + K_1 + 1$, $C \neq 2K_1 + 2K_2 + 1$, $M > K_1 > 1$ and $\bar{d}(u, v) = K_1$.*

Note that we are only interested in distances not equal to M.

Proof. Suppose that the statement is not true, and let $\mathbf{G}' = (G, d')$ be a counterexample. Recall that the completion with a magic parameter M is defined as a limit of a sequence $\mathbf{G}_1, \mathbf{G}_2, \dots$ of edge-labelled graphs such that $\mathbf{G}_1 = \mathbf{G}$ and each two subsequent edge-labelled graphs differ at most by adding edges of a single distance.

Take the smallest i such that in $\mathbf{G}_i = (G, d_i)$ there are vertices $u, v \in G$ with $d(u, v) = d_i(u, v)$ and $d'(u, v)$ not satisfying the lemma. Denote by w a witness of the distance $d_i(u, v)$. As in the proof Lemma 4.6, we can argue that $d_{i-1}(u, w)$ respectively $d_{i-1}(v, w)$ satisfy the optimality conditions 1 or 2 in Lemma 4.6.

First we will show that the exceptional case 2 from the statement only happens at the very end of the induction, hence when using the induction hypothesis (or minimality of i), we can work only with the first part of the statement.

Suppose that the parameters satisfy Case (III) and further $C = 2\delta + K_1 + 1$, $C \neq 2K_1 + 2K_2 + 1$ and $M > K_1 > 1$. We have $t_M(K_1) > t_M(a)$ for any distance $a < K_1$ and also, by the admissibility, $t_M(K_1) > t_M(b)$ for any distance $b \geq K_2$ and $b > M$: since $t_M(K_1) = 2K_1 - 1$ and $t_M(b) \leq 2\delta - 2K_2$, we need to verify that $2K_1 - 1 > 2\delta - 2K_2$ and thus $2K_1 + 2K_2 > 2\delta + 1$. By admissibility it follows $2K_2 + K_1 \geq 2\delta$ (when $2K_2 + K_1 = 2\delta - 1$, admissibility implies $C \geq 2\delta + K_1 + 2$), which give the desired bound.

Next observe that if $K_1 = 1$ then from Lemma 4.6 we have that whenever $d(u, v) = 1$ for some vertices u, v, then in any completion the edge has also length 1, hence also fixed parity.

As in the proof of Lemma 4.6, we will now distinguish three cases based on whether $d_i(u, v)$ was introduced due to $\mathcal{F}^+$, $\mathcal{F}^-$ or $\mathcal{F}^C$:

$\mathcal{F}^+$ case. In this case $d_i(u, v) < M$ and $d_i(u, v) = d_{i-1}(u, w) + d_{i-1}(w, v)$. Because of our assumption $d_i(u, v) \leq K_1$, the perimeter of the triangle uvw in $\mathbf{G}_i$ is even and at most $2K_1$. By Lemma 4.6 we have $d'(u, v) \leq d_i(u, v)$, $d'(u, w) \leq d_{i-1}(u, w)$ and $d'(v, w) \leq d_{i-1}(v, w)$, hence $d'(u, v) + d'(u, w) + d'(w, v)$ is odd and smaller than $2K_1 + 1$. Thus the triangle uvw is forbidden by the K_1 bound in $\mathbf{G}'$, a contradiction.

$\mathcal{F}^-$ case. Here $d_i(u, v) > M$ and (without loss of generality) $d_i(u, v) = d_{i-1}(u, w) - d_{i-1}(w, v)$. Then the triangle uvw has even perimeter with respect to d_i. By our assumption we have $d_i(u, v) \geq K_2$ and thus $d_{i-1}(u, w) > d_i(u, v) \geq K_2$ and $d_{i-1}(v, w) < M$.

This implies $d_i(u, v) + d_{i-1}(u, w) = 2d_i(u, v) + d_{i-1}(v, w) \geq 2K_2 + d_{i-1}(v, w)$. From Lemma 4.6 we get that $d'(u, v) \geq d_i(u, v)$, $d'(u, w) \geq d_{i-1}(u, w)$ and $d'(v, w) \leq d_{i-1}(v, w)$, hence also $d'(u, v) + d'(u, w) \geq 2K_2 + d'(v, w)$ holds. Thus the triangle uvw is forbidden by the K_2 bound in $\mathbf{G}'$, a contradiction.

$\mathcal{F}^C$ case. Here $d_i(u, v) < M$ and $d_i(u, v) = C - 1 - d_{i-1}(u, w) - d_{i-1}(w, v)$. From our assumption it follows that $d_i(u, v) \leq K_1$.

In Case (III) we have $C \geq 2\delta + K_1 + 1$, hence $d_i(u, v) = K_1$ if and only if $C = 2\delta + K_1 + 1$, $M > K_1$ and $d(u, w) = d(v, w) = \delta$; this case is treated in point 2.

It remains to consider Case (II). Hence we can assume that $d'(u, v) < d(u, v)$ and these edges have different parity. Note that then the triangle uvw has even perimeter $C - 1$. By Lemma 4.6 we have $d'(u, w) +$

$d'(v,w) \geq d_{i-1}(u,w) + d_{i-1}(v,w) = C - 1 - d(u,v) = 2K_1 + 2K_2 - d_i(u,v)$. But as $d'(u,v) \leq d_i(u,v) \leq K_1$ we have $d'(u,w) + d'(v,w) \geq 2K_2 + d'(u,v)$, so the triangle uvw is forbidden by the K_2 bound in $\mathbf{G}'$, a contradiction. □

Lemma 4.8 (Automorphism Preservation Lemma)**.** *Let $\mathbf{G} \in \mathcal{G}^\delta$ and let $\overline{\mathbf{G}}$ be its completion with a magic parameter M. Then every automorphism of $\mathbf{G}$ is also an automorphism of $\overline{\mathbf{G}}$.*

Proof. Given $\mathbf{G}$ and an automorphism $f : \mathbf{G} \to \mathbf{G}$, it can be verified by induction that for every $k > 0$, f is also an automorphism graph $\mathbf{G}_k$ as in Definition 4.2. For every edge x, y of $\mathbf{G}_k$ which is not an edge of $\mathbf{G}_{k-1}$, it is true that $f(x), f(y)$ is also an edge of $\mathbf{G}_k$ which is not an edge of $\mathbf{G}_{k-1}$, and moreover the edges x, y and $f(x), f(y)$ are of the same length. This follows directly from the definition of $\mathbf{G}_k$. □

4.4. *Correctness of the completion algorithm*

In this section we prove:

Theorem 4.9. *Let δ, K_1, K_2, C_0 and C_1 be primitive admissible parameters. Suppose that $\mathbf{G} = (G, d) \in \mathcal{G}^\delta$ has a completion into $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$, and let $\overline{\mathbf{G}} = (G, \bar{d})$ be its completion with a magic parameter M. Then $\overline{\mathbf{G}} \in \mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$.*

$\overline{\mathbf{G}}$ is optimal in the following sense: Let $\mathbf{G}' = (G, d') \in \mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ be an arbitrary completion of $\mathbf{G}$ in $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$, then for every pair of vertices $u, v \in G$ one of the following holds:

1. *$d'(u,v) \geq \bar{d}(u,v) \geq M$,*
2. *$d'(u,v) \leq \bar{d}(u,v) \leq M$,*
3. *the parameters δ, K_1, K_2, C_0 and C_1 satisfy Case (IIB), $d'(u,v) \neq M$ and $\bar{d}(u,v) = M - 1$.*

Finally, every automorphism of $\mathbf{G}$ is also an automorphism of $\overline{\mathbf{G}}$.

In the next five lemmas we will use Lemmas 4.6 and 4.7 to show that $\mathbf{G} \in \mathcal{G}^\delta$ has a completion into $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$, if and only if the algorithm with a magic parameter M yields such a completion. We will deal with each type of forbidden triangle separately, and in doing that, we will implicitly use the results of Section 4.2.

Lemma 4.10 (C-bound Lemma)**.** *Suppose $C' = C + 1$, and let $\mathbf{G} = (G, d) \in \mathcal{G}^\delta$ be such that there is a completion of $\mathbf{G}$ into $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$; let $\overline{\mathbf{G}} = (G, \bar{d})$ be its completion with a magic parameter M. Then there is no triangle forbidden by the C bound in $\overline{\mathbf{G}}$.*

Proof. Suppose for a contradiction that there is a triangle with vertices u, v, w in $\overline{\mathbf{G}}$ such that $\bar{d}(u,v) + \bar{d}(v,w) + \bar{d}(u,w) \geq C$. For brevity let $a = \bar{d}(u,v)$, $b = \bar{d}(v,w)$ and $c = \bar{d}(u,w)$. Assume without loss of generality that $a \leq b \leq c$. Let a', b', c' be the corresponding edge lengths in an arbitrary completion of $\mathbf{G}$ into $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$. Then two cases can appear.

Either $a, b, c > M$, and then by Lemma 4.6 we have $a' \geq a$, $b' \geq b$ and $c' \geq c$, so we get the contradiction $a' + b' + c' \geq C$; or $a \leq M$, $c \geq b > M$ and $a + b + c \geq C$. In this case Lemma 4.6 implies $b' \geq b$ and $c' \geq c$ and $a' \leq a$. If the edge (u,v) was already in $\mathbf{G}$, then clearly $a' + b' + c' \geq a + b + c \geq C$, which is a contradiction. If (u,v) was not already an edge in $\mathbf{G}$, then it was added by the completion algorithm with the magic parameter M in step $t_M(a)$. Let $\bar{a} = C - 1 - b - c$. Then clearly $\bar{a} < a$, which means that $t_M(\bar{a}) < t_M(a)$, and as $\bar{a}$ expands to b, c, we have $t_M(b), t_M(c) < t_M(\bar{a})$. But then the completion with the magic parameter M actually sets the length of the edge u, v to be $\bar{a}$ in step $t_M(\bar{a})$, which is a contradiction. □

Lemma 4.11 (Metric Lemma)**.** *Let $\mathbf{G}$ and $\overline{\mathbf{G}}$ be as in Lemma 4.10. Then there are no non-metric triangles in $\mathbf{G}$.*

Proof. Suppose for a contradiction that there is a triangle with vertices u, v, w in $\mathbf{G}$ such that $d(u,v) + d(v,w) < d(u,w)$. Denote $a = d(u,v)$, $b = d(v,w)$ and $c = d(u,w)$ and assume without loss of generality that $a \leq b < c$. Let a', b', c' be the corresponding edge lengths in an arbitrary completion of $\bar{\mathbf{G}}$ into $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$. We shall distinguish three cases based on Section 4.2:

1. First suppose $a, b, c < M$. Then $t_M(a) \leq t_M(b) < t_M(a+b) < t_M(c)$, which means that c must be already in $\mathbf{G}$. Note that in Case (IIB) if $b = K_1 - 1 = M - 1$, then $c \geq M$, hence we can use Lemma 4.6 for a and b, which gives us that $a' + b' \leq a + b < c = c'$, which is a contradiction.

2. Another possibility is $a < M$ and $b, c \geq M$ (actually $c > M$, since abc is non-metric).

   Suppose $a' \leq a$ and $c' \geq c$ (the first possibility of Lemma 4.6). If b was already in $\mathbf{G}$, then $\mathbf{G}$ has no completion – a contradiction. Otherwise clearly $c - a > b \geq M$, so $t_M(c-a) < t_M(b)$. But as $c - a$ expands to c and a, we get $t_M(c-a) > t_M(c), t_M(a)$, which means that the completion algorithm with the magic parameter M would complete the edge v, w with the length $c - a$ and not with b.

   If the previous paragraph does not apply we have Case (IIB) and $a = K_1 - 1 = K_2 - 1$. But then as $M = K_2$, we have $b \geq K_2$, which means $a + b \geq 2K_2 - 1 = \frac{4\delta-2}{3} - 1 \geq \delta$ for $\delta \geq 5$, which holds in (IIB), but that means that abc is actually metric, a contradiction.

3. The last possibility is $a, b < M$ and $c \geq M$. Then either (by Lemma 4.6 and Lemma 4.4 if $c = M$) we have $a' \leq a$, $b' \leq b$ and $c' \geq c$, hence the triangle a', b', c' is again non-metric, or we have Case (IIB), $b = K_1 - 1$, $a \leq K_1 - 1$. The rest of proof of this lemma consists of a verification of this special case.

From admissibility of (IIB) we have $M = K_1 = K_2 = \frac{2\delta-1}{3}$ and $\delta \geq 5$. Note that $c - a \geq b + 1 = K_1 = M$ from non-metricity of abc, hence $c > M$.

If both a and b were already in $\bar{\mathbf{G}}$, then abc is non-metric in any completion by Lemma 4.6. The same thing is true if b was already in $\bar{\mathbf{G}}$ and $a' \leq a$ in any completion (i.e. either $a < K_1 - 1$ or a was not introduced by $\mathcal{F}^C$ due to a (δ, δ) fork).

Note that for $\delta \geq 8$ it cannot happen that $a = b = K_1 - 1$, as then $a + b = 2K_1 - 2 = 2\frac{2\delta-1}{3} - 2 \geq \delta$, hence $a + b < c$ is absurd. So the only case when $a = b = K_1 - 1$ is $\delta = 5$ (because from (IIB) it follows that $\delta = 3m + 2$ for some $m \geq 1$). In that case we have triangle $5, 2, 2$ and each of the twos either was in $\bar{\mathbf{G}}$ or is supported by a fork $(1, 1)$ or by a fork $(5, 5)$. And it can be shown that none of these structures has a strong completion into $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$.

Hence b was not in the input graph and $a < K_1 - 1$.

Observe that $c - a = M$. From non-metricity of abc we have $c - a \geq b + 1 = K_1 = M$. And if $c - a \geq M + 1$, then $t_M(c-a) \leq t_M(M+1) = 2\delta - 2M - 2$. And this is strictly less than $t_M(M-1) = 2M - 3$ since $M = \frac{2\delta-1}{3}$ and $\delta \geq 5$. Further as $M = \frac{2\delta-1}{3}$ is odd, we see that a and c have different parities.

From Lemma 4.6 we have that in any completion $c' \geq c$ and $a' \leq a$. So the only way that the triangle u, v, w can be metric is to have $b' > b$. Note that $c' - a' \geq c - a = M = K_2$, hence $c' \geq a' + K_2$. And from Lemma 4.7 we have that the parities of a, b, c are preserved.

Note that as M is odd, b' is even. And since the parities of c' and a' are different, we have that $a' + b' + c'$ is odd. Also note that $c' + b' \geq a' + K_2 + K_2 + 1 \geq 2K_2 + a'$. Hence u, v, w is forbidden by the K_2 bound in $\mathbf{G}'$, which is a contradiction. □

Lemma 4.12 (K_1-bound Lemma)**.** *Let $\mathbf{G}$, $\overline{\mathbf{G}}$ be as in Lemma 4.10. Then there are no triangles forbidden by the K_1-bound in $\overline{\mathbf{G}}$.*

Proof. Suppose for a contradiction that there is a metric (from Lemma 4.11 we already know that all triangles in $\mathbf{G}$ are metric) triangle with vertices u, v, w in $\mathbf{G}$ such that $\bar{d}(u,v) + \bar{d}(v,w) + \bar{d}(u,w)$ is odd and less than $2K_1 + 1$. Denote $a = \bar{d}(u,v)$, $b = \bar{d}(v,w)$ and $c = \bar{d}(u,w)$. From Section 4.2 we get $a, b, c < K_1 \leq M$.

First suppose that Lemma 4.6 gives us that for any completion a', b', c' that $a' \leq a$, $b' \leq b$ and $c' \leq c$. Also a has the same parity as a', b as b' and c as c' by Lemma 4.7, hence $a' + b' + c' \leq a + b + c$ and those two expressions have the same parity, hence a', b', c' is also forbidden by the K_1 bound, a contradiction.

Otherwise we have Case (IIB) and $c = K_1 - 1$. But then from metricity of abc either $a + b = c$ (but then $a + b + c$ is even, a contradiction), or $a + b = c + 1$ (if $a + b \geq c + 2$, then the perimeter of the triangle is too large to be forbidden by the K_1 bound). But again in any completion $a' \leq a$ and $b' \leq b$, so either $c' \leq c$ or $c' = c + 1$ (from metricity). From Lemma 4.7 we know that the parity of c is preserved, hence $c' = c + 1$ is absurd, so $a' \leq a$, $b' \leq b$ and $c' \leq c$, and we can apply the same argument as in the preceding paragraph. □

Lemma 4.13 (C_0, C_1-bound Lemma)**.** *Let $C' > C + 1$ and let* $\mathbf{G}$*,* $\overline{\mathbf{G}}$ *be as in Lemma 4.10. Then there are no triangle forbidden by either the C_0 or C_1 bounds in* $\overline{\mathbf{G}}$*.*

Proof. Suppose for a contradiction that there is a triangle with vertices u, v, w in $\overline{\mathbf{G}}$, such that $a+b+c \geq C$ and has parity such that it is forbidden by one of the C bounds, where $a = \bar{d}(u, v)$, $b = \bar{d}(v, w)$ and $c = \bar{d}(u, w)$.

In Case (IIB), we have $K_1 = K_2$, $C = 2K_1 + 2K_2 + 1 = 4K_2 + 1$ and $3K_2 = 2\delta - 1$, hence $C = 2\delta + K_2$. For parameters from Case (III) we have $C \geq 2\delta + K_2$, which means that we always have $C \geq 2\delta + K_2$. This implies that $b, c > K_2 \geq M$ and $a \geq K_2$. If a was already present in $\mathbf{G}$, then by Lemmas 4.4, 4.6 and 4.7 we have that any completion a', b', c' has $a' = a$, $b' \geq b$ and $c' \geq c$ and the parities are preserved, hence a', b', c' is forbidden by the C bound as well, a contradiction to $\mathbf{G}$ having a completion. If a is not in $\mathbf{G}$, we have $a \neq M$ (by Lemma 4.4) and actually $a > M$ as $a \geq K_2 \geq M$. Thus we can again use Lemmas 4.6 and 4.7 to get a contradiction. □

Lemma 4.14 (K_2-bound Lemma)**.** *Let* $\mathbf{G}$*,* $\overline{\mathbf{G}}$ *be as in Lemma 4.10. Then there are no triangles forbidden by the K_2-bound in* $\overline{\mathbf{G}}$*.*

Proof. We know that all triangles are metric and not forbidden by the C bounds. Suppose for a contradiction that there is a triangle with vertices u, v, w in $\overline{\mathbf{G}}$ such that $b + c \geq 2K_2 + a + 1$, where $a = \bar{d}(u, v)$, $b = \bar{d}(v, w)$ and $c = \bar{d}(u, w)$. We know that $b, c > K_2$ and $a \leq K_1$ by Section 4.2, where equality can occur only in Case (III) when $2K_2 + K_1 = 2\delta - 1$ and furthermore $M > a$ (because of Definition 4.4).

Note that from the conditions for Case (III), we know that if $2K_2 + K_1 = 2\delta - 1$, then $C \geq 2\delta + K_1 + 2$, which means that for edges a, b, c Lemma 4.7 guarantees that the parity is preserved.

Unless $a = K_1 - 1$, Case (IIB), Lemmas 4.6 and 4.7 yield that $a' + b' + c'$ has the same parity as $a + b + c$ for any completion a', b', c' and $b' \geq b$, $c' \geq c$ and $a' \leq a$, hence triangle u, v, w is forbidden by the K_2 bound in any completion of $\mathbf{G}$, which is a contradiction.

The last case remaining is $a = K_1 - 1 = K_2 - 1$, Case (IIB). But then $b + c \geq 2K_2 + a + 1 = 3K_2 = 2\delta - 1$, so either $b + c = 2\delta - 1$, or $b + c = 2\delta$. But from being forbidden by the K_2-bound we know that $a + b + c$ is odd, hence $b + c$ has different parity than a. And we know that $a = K_2 - 1 = \frac{2\delta - 1}{3} - 1$, which is even, hence $b + c = 2\delta - 1$. We also know that parities are preserved, so if $a' \geq K_2 + 1$, then $a' + b' + c' \geq 2\delta + K_2$ and it is thus forbidden by the C bound. □

Proof of Theorem 4.9. From the Lemmas 4.10, 4.11, 4.12, 4.13, 4.14 we conclude that the algorithm will correctly complete every graph $\mathbf{G}$ which has completion into $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$. The optimality statement follows by Lemma 4.6. Automorphisms are preserved according to Lemma 4.8. □

4.5. Stationary independence relation

In this section we show a corollary to Theorem 4.9 proving that the completion with a magic parameter M has the right properties needed to define a stationary independence relation.

As pointed out in [28], certain "canonical" ways of amalgamation give rise to stationary independence relations. We are going to show that also the opposite holds and every stationary independence relation on a homogeneous structure induces a canonical amalgamation operator. We start with some useful observations on stationary independence relations.

Lemma 4.15. *Let $\mathop{\raise0.2ex\hbox{$\smile$}}\!\!\!\!\!\!\mid\,$ be a (local) SIR. Then the following properties hold:*

1. $\mathbf{A} \mathop{\perp}_{\mathbf{C}} \mathbf{B} \leftrightarrow \langle \mathbf{AC} \rangle \mathop{\perp}_{\mathbf{C}} \mathbf{B} \leftrightarrow \mathbf{A} \mathop{\perp}_{\mathbf{C}} \langle \mathbf{BC} \rangle$
2. (Transitivity). $\mathbf{A} \mathop{\perp}_{\mathbf{C}} \mathbf{B} \wedge \mathbf{A} \mathop{\perp}_{\langle \mathbf{BC} \rangle} \mathbf{D} \rightarrow \mathbf{A} \mathop{\perp}_{\mathbf{C}} \mathbf{D}$
3. $\mathbf{A} \mathop{\perp}_{\mathbf{C}} \langle \mathbf{BD} \rangle \leftrightarrow \mathbf{A} \mathop{\perp}_{\mathbf{C}} \mathbf{B} \wedge \mathbf{A} \mathop{\perp}_{\langle \mathbf{BC} \rangle} \mathbf{D}$

Proof. Point (1) and (2) are shown in [28] and [29] respectively. In order to show (3), i.e. that the inverse direction of the implication in the Monotonicity axiom holds, note that by (1) we have $\mathbf{A} \mathop{\perp}_{\langle \mathbf{BC} \rangle} \mathbf{D} \leftrightarrow \mathbf{A} \mathop{\perp}_{\langle \mathbf{BC} \rangle} \langle \mathbf{BD} \rangle$. Then, by (2) $\mathbf{A} \mathop{\perp}_{\mathbf{C}} \mathbf{B} \wedge \mathbf{A} \mathop{\perp}_{\langle \mathbf{BC} \rangle} \langle \mathbf{BD} \rangle \rightarrow \mathbf{A} \mathop{\perp}_{\mathbf{C}} \langle \mathbf{BD} \rangle$. □

Definition 4.5. Let C be an amalgamation class. Let us say that $\oplus$ is an *amalgamation operator*, if it assigns to every triple of structures $\mathbf{A}, \mathbf{B_1}, \mathbf{B_2} \in C$ with embeddings $e_1 : \mathbf{A} \to \mathbf{B_1}$ and $e_2 : \mathbf{A} \to \mathbf{B_2}$ a unique amalgam, i.e. a structure $\mathbf{D} \in C$ and embeddings $f_1 : \mathbf{B_1} \to \mathbf{D}$, $f_2 : \mathbf{B_2} \to \mathbf{D}$, such that $f_1 \circ e_1 = f_2 \circ e_2$. For short let us then write $\mathbf{D} = \mathbf{B_1} \oplus_{\mathbf{A}} \mathbf{B_2}$. Let us call $\oplus$ a *local* amalgamation operator if it is only defined for non-empty $\mathbf{A}$. Let us call $\oplus$ is *canonical* on C, if additionally the following hold:

1. $\mathbf{B_1} \oplus_{\mathbf{A}} \mathbf{B_2}$ has minimal domain, i.e. it is generated by the union of $f_1(\mathbf{B_1})$ and $f_2(\mathbf{B_2})$
2. Monotonicity: If $\mathbf{B_1} \oplus_{\mathbf{A}} \mathbf{B_2} = \langle f_1(\mathbf{B_1}) \cup f_2(\mathbf{B_2})\rangle$, then $\mathbf{B_1} \oplus_{\mathbf{A}} \mathbf{B_2'} = \langle f_1(\mathbf{B_1}) \cup f_2(\mathbf{B_2'})\rangle$ for all substructures $e_2(\mathbf{A}) \subseteq \mathbf{B_2'} \subseteq \mathbf{B_2}$
3. Associativity: $(\mathbf{A} \oplus_{\mathbf{C_1}} \mathbf{B}) \oplus_{\mathbf{C_2}} \mathbf{D} = \mathbf{A} \oplus_{\mathbf{C_1}} (\mathbf{B} \oplus_{\mathbf{C_2}} \mathbf{D})$.

Then the following holds:

Theorem 4.16. *A homogeneous structure* $\mathbf{M}$ *admits a (local) stationary independence relation if and only if* $\operatorname{Age}(\mathbf{M})$ *has a (local) canonical amalgamation operator. Moreover there is a 1-to-1 correspondence between (local) stationary independence relations* $\perp$ *and (local) canonical amalgamation operators* $\oplus$ *by:* $\mathbf{A} \perp_{\mathbf{C}} \mathbf{B}$ *if and only if* $\langle \mathbf{ABC}\rangle$ *is isomorphic to* $\langle \mathbf{AC}\rangle \oplus_{\mathbf{C}} \langle \mathbf{BC}\rangle$.

Proof. We first show that every canonical amalgamation operator gives rise to a stationary independence relation. Examples of this fact were already given in [28]. Let $\oplus$ be a canonical amalgamation operator on $\operatorname{Age}(\mathbf{M})$. Then we define a stationary independence relation by setting $\mathbf{A} \perp_{\mathbf{C}} \mathbf{B}$ if and only if $\langle \mathbf{ABC}\rangle$ is isomorphic to $\langle \mathbf{AC}\rangle \oplus_{\mathbf{C}} \langle \mathbf{BC}\rangle$ under an isomorphism commuting with the embeddings. The axioms SIR1, SIR2, SIR4 and SIR5 follow straightforwardly from the fact that $\oplus$ is an amalgamation operator.

For (SIR3), observe first that by the minimality of $\oplus$ we have that $\langle \mathbf{XY}\rangle = \mathbf{X} \oplus_{\mathbf{X}} \langle \mathbf{XY}\rangle$ for all $\mathbf{X}, \langle \mathbf{XY}\rangle \in \operatorname{Age}(\mathbf{M})$. Let $\mathbf{A} \perp_{\mathbf{C}} \langle \mathbf{BD}\rangle$; by our observation this is equivalent to

$$\langle \mathbf{ABDC}\rangle = \langle \mathbf{AC}\rangle \oplus_{\mathbf{C}} \langle \mathbf{BDC}\rangle = \langle \mathbf{AC}\rangle \oplus_{\mathbf{C}} (\langle \mathbf{BC}\rangle \oplus_{\langle \mathbf{BC}\rangle} \langle \mathbf{BCD}\rangle).$$

Since $\oplus$ is associative, this is equivalent to $\langle \mathbf{ABDC}\rangle = (\langle \mathbf{AC}\rangle \oplus_{\mathbf{C}} \langle \mathbf{BC}\rangle) \oplus_{\langle \mathbf{BC}\rangle} \langle \mathbf{BCD}\rangle$. Since $\oplus$ is monotone, this implies $(\langle \mathbf{AC}\rangle \oplus_{\mathbf{C}} \langle \mathbf{BC}\rangle) = \langle \mathbf{ABC}\rangle$, hence $\mathbf{A} \perp_{\mathbf{C}} \mathbf{B}$ and $\mathbf{A} \perp_{\langle \mathbf{BC}\rangle} \mathbf{D}$. This concludes the proof that $\perp$ is a SIR.

For the opposite direction, let $\perp$ be a stationary independence relation on the substructures of $\mathbf{M}$. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be in the age of $\mathbf{M}$ and let $e_1 : \mathbf{C} \to \mathbf{A}$ and $e_2 : \mathbf{C} \to \mathbf{B}$ be embeddings. By the homogeneity of $\mathbf{M}$ there are embeddings $f_1 : \mathbf{A} \to \mathbf{M}$ and $f_2 : \mathbf{B} \to \mathbf{M}$ such that $f_1 e_1 = f_2 e_2$. By SIR4, there is an $\mathbf{A}'$ such that $\mathbf{A}' \perp_{f_1 e_1(\mathbf{C})} f_2(\mathbf{B})$ and such that there is an automorphism α of $\mathbf{M}$ fixing $f_1 e_1(\mathbf{C})$ that maps $f_1(\mathbf{A})$ to $\mathbf{A}'$. We then define $\mathbf{A} \oplus_{\mathbf{C}} \mathbf{B}$ as the amalgam $\langle \mathbf{A}' \cup f_2(\mathbf{B})\rangle$ with respect to the embeddings αf_1 and f_2. By definition $\oplus$ is an amalgamation operator, such that $\mathbf{A} \oplus_{\mathbf{C}} \mathbf{B}$ has minimal domain.

For showing the associativity, let $\mathbf{A}'$ and $\mathbf{B}'$ be such that $\mathbf{B} \oplus_{\mathbf{C_2}} \mathbf{D} = \langle \mathbf{B}'\mathbf{D}\rangle$ and $\mathbf{A} \oplus_{\mathbf{C_1}} (\mathbf{B} \oplus_{\mathbf{C_2}} \mathbf{D}) = \langle \mathbf{A}'\mathbf{B}'\mathbf{D}\rangle$. Thus we have $\mathbf{B}' \perp_{\mathbf{C_2}} \mathbf{D}$ and $\mathbf{A}' \perp_{\mathbf{C_1}} \langle \mathbf{B}'\mathbf{D}\rangle$. By Lemma 4.15 (3) this is equivalent to

$$\mathbf{A}' \mathop{\perp}_{\mathbf{C_1}} \mathbf{B}' \wedge \mathbf{A}' \mathop{\perp}_{\mathbf{B}'} \mathbf{D} \wedge \mathbf{B}' \mathop{\perp}_{\mathbf{C_2}} \mathbf{D}.$$

Again by Lemma 4.15 (3) and $\mathbf{C_2} \subseteq \mathbf{B}'$ this is equivalent to

$$\mathbf{A}' \mathop{\perp}_{\mathbf{C_1}} \mathbf{B}' \wedge \mathbf{D} \mathop{\perp}_{\mathbf{C_2}} \langle \mathbf{A}'\mathbf{B}'\rangle.$$

By our definition of $\oplus$ we then have

$$\langle \mathbf{A}'\mathbf{B}'\mathbf{D}\rangle = \mathbf{A} \oplus_{\mathbf{C_1}} (\mathbf{B} \oplus_{\mathbf{C_2}} \mathbf{D}) = (\mathbf{A} \oplus_{\mathbf{C_1}} \mathbf{B}) \oplus_{\mathbf{C_2}} \mathbf{D},$$

proving that $\oplus$ is associative. □

Corollary 4.17. *Let* δ*,* K_1*,* K_2*,* C_0 *and* C_1 *be primitive admissible parameters. For every magic parameter* M *we can define a* stationary independence relation with magic parameter M *on* $\Gamma^{\delta}_{K_1,K_2,C_0,C_1}$ *as follows:* $\mathbf{A} \perp^{M}_{\mathbf{C}} \mathbf{B}$ *if and only if* $\langle \mathbf{ABC}\rangle$ *is isomorphic to the completion with magic parameter* M *of the free amalgamation of* $\langle \mathbf{AC}\rangle$ *and* $\langle \mathbf{BC}\rangle$ *over* $\mathbf{C}$.

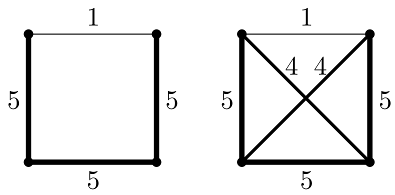

Figure 6: Completing the cycle 1555.

Proof. For structures $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$, such that $\mathbf{C}$ embeds into $\mathbf{A}$ and $\mathbf{B}$ we define $\mathbf{A} \oplus_{\mathbf{C}} \mathbf{B}$ to be the completion with the magic parameter M of the free amalgam of $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{C}$. If we can show that $\oplus$ is a canonical amalgamation operator, then we are done by Theorem 4.16. By Theorem 4.9, $\mathbf{A} \oplus_{\mathbf{C}} \mathbf{B}$ is an element of $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$, hence $\oplus$ is an amalgamation operator. The monotonicity and associativity of $\oplus$ follow straightforward from the optimality property of the completion with the magic parameter M (Theorem 4.9 (1),(2)). $\square$

4.6. *Ramsey property and EPPA*

Theorem 4.9 implies the following two lemmas that are crucial when applying Theorems 2.2 and 2.1.

Lemma 4.18 (Finite Obstacles Lemma)**.** *Let δ, K_1, K_2, C_0 and C_1 be primitive admissible parameters. Then the class $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ has a finite set of obstacles which are all cycles of diameter at most $2^{\delta}3$.*

Example. Consider $\mathcal{A}^{5}_{3,3,16,13}$ discussed in Section 4.1. The set of obstacles of this class contains all the forbidden triangles listed earlier, but in addition to that it also contains some cycles with 4 or more vertices. A complete list of those can be obtained by running the algorithm backwards from the forbidden triangles.

All such cycles with 4 vertices can be constructed from the triangles by substituting distances by the forks depicted at Figure 4. This means substituting 2 for 11 or 55, and 4 for 15 or 51. With equivalent cycles removed this give the following list:

$$\begin{array}{rccl}
\text{non-metric:} & 124 & \Longrightarrow & 1\mathbf{11}4, 1\mathbf{55}4, 12\mathbf{15}, 12\mathbf{51} \\
 & 125 & \Longrightarrow & 1\mathbf{11}5^{*}, 1\mathbf{55}5^{**} \\
 & 114 & \Longrightarrow & 11\mathbf{15} \\
 & 225 & \Longrightarrow & \mathbf{11}25^{*}, \mathbf{55}25 \\
K_1\text{-bound:} & 122 & \Longrightarrow & 1\mathbf{11}2, 1\mathbf{55}2^{*} \\
K_2\text{-bound:} & 144 & \Longrightarrow & 1\mathbf{15}4, 1\mathbf{51}4, 14\mathbf{15} \\
 & 245 & \Longrightarrow & \mathbf{11}45, \mathbf{55}45, 2\mathbf{15}5, 25\mathbf{15} \\
C\text{-bound:} & 445 & \Longrightarrow & \mathbf{15}45, \mathbf{51}45, 4\mathbf{15}5
\end{array}$$

Observe that running the algorithm may produce multiple forbidden triangles which leads to duplicated cycles in the list. Such duplicates are denoted by $*$. For example, 125 was expanded to $1\mathbf{11}5$. The algorithm will first notice the fork $(1, 5)$ and produce 114. This is also a forbidden triangle, but a different one. In the case of $1\mathbf{55}5$ (another expansion of 125) the algorithm will again use the fork $(1, 5)$ first and produce the triangles 455 and 145, which are valid triangles, see Figure 6. Not all expansions here are necessarily forbidden, because not all of them correspond to a valid run of the algorithm. However with the exception of cases denoted by $**$ all the above 4-cycles are forbidden.

Repeating the procedure one obtains the following cycles with five edges that cannot be completed into this class of metric graphs:

$$11111, 11115, 11155, 11515, 15155, 11555, 15555, 55555.$$

An example of failed run of algorithm trying to complete one of the forbidden cycles is depicted at Figure 7. Because there are no distances of 2 or 4 in the cycles with five edges that cannot be completed into this class, it follows that all cycles with at least six edges can be completed.

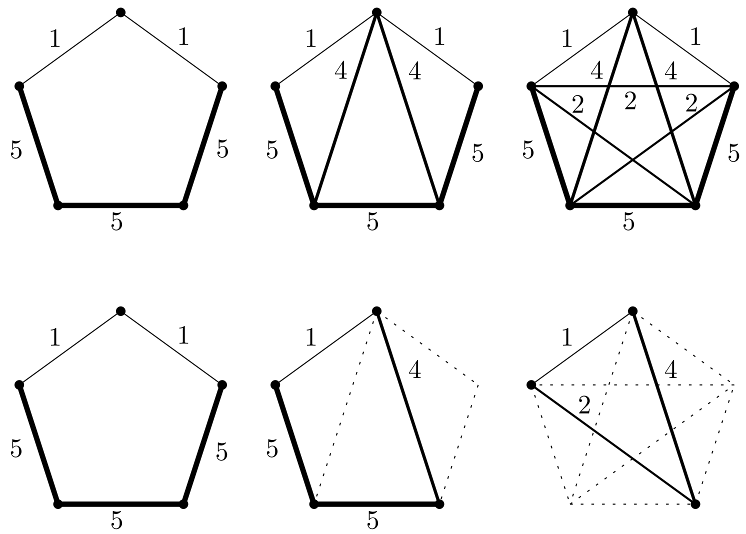

Figure 7: Failed run attempting to complete cycle 11555. In the bottom row is the backward run from non-metric triangle 124 to the original obstacle used in the proof of Lemma 4.18.

Proof of Lemma 4.18. Let $\mathbf{G} = (G, d) \in \mathcal{G}^\delta$ be an edge-labelled graph no completion in $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$. We seek a subgraph of $\mathbf{G}$ of bounded size which has also no completion into $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$.

Consider the sequence of graphs $\mathbf{G}_0, \mathbf{G}_1, \ldots, \mathbf{G}_{2M+1}$ as given by Definition 4.4 when completing $\mathbf{G}$ with magic parameter M. Set $\mathbf{G}_{2M+2}$ to be the actual completion.

Because $\mathbf{G}_{2M+2} \notin \mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ we know it contains a forbidden triangle $\mathbf{O}$. This triangle always exists, because $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ is 3-constrained. By backward induction on $k = 2M+1, 2M, \ldots, 0$ we obtain cycles $\mathbf{O}_k$ of $\mathbf{G}_k$ such that $\mathbf{O}_k$ has no completion in $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ and there exists a homomorphism $f : \mathbf{O}_k \to \mathbf{G}_k$.

Put $\mathbf{O}_{2M+1} = \mathbf{O}$. By Lemma 4.4 we know that this triangle is also in $\mathbf{G}_{2M+1}$. At step k consider every edge u, v of $\mathbf{O}_{k+1}$ which is not an edge of $\mathbf{G}_k$ considering its witness w (i.e. vertex w such that the edges u, w and v, w implied the addition of the edge u, v) and extending $\mathbf{O}_k$ by a new vertex w' and edges $d(u, w') = d(u, w)$ and $d(v, w') = d(v, w)$. One can verify that the completion algorithm will fail to complete $\mathbf{O}_k$ the same way as it failed to complete $\mathbf{O}_{k+1}$ and moreover there is a homomorphism $\mathbf{O}_{k+1} \to \mathbf{G}_k$.

At the end of this procedure we obtain $\mathbf{O}_0$, a subgraph of $\mathbf{G}$, that has no completion into $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$. The bound on the size of the cycle follows from the fact that only δ steps of the algorithm are actually changing the graph and each time every edge may introduce at most one additional vertex.

Let $\mathcal{O}$ consist of all edge-labelled cycles with at most $2^\delta 3$ vertices that are not completable in $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$. Clearly $\mathcal{O}$ is finite. To check that $\mathcal{O}$ is a set of obstacles it remains to verify that there is no $\mathbf{O} \in \mathcal{O}$ with a homomorphism to some $\mathbf{M} \in \mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$. Denote by $\mathcal{O}'$ the set of all homomorphic images of structures in $\mathcal{O}$. Assume, to the contrary, the existence of such an $\mathbf{O} = (O, d) \in \mathcal{O}'$ and $\mathbf{M} = (M, d')$ and a homomorphism $f : \mathbf{O} \to \mathbf{M}$ and among all those choose one minimising $|\mathbf{O}|$. It follows that $|\mathbf{O}| - |\mathbf{M}| = 1$. Denote by x, y the pair of vertices identified by f. Let $\mathbf{O}' = (O, d'')$ be a metric graph such that $d''(z, z') = d(f(z), f(z'))$ for every pair $\{z, z'\} \neq \{x, x'\}$. It follows that because $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ has the strong amalgamation property, and also $\mathbf{O}' = (O, d'')$ has a completion in $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$. □

To demonstrate the use of these results, we show the following theorems which we later strengthen in Section 3.2 for classes with Henson constraints.

Theorem 4.19. *For every choice of admissible primitive parameters δ, K_1, K_2, C_0 and C_1, the class $\overrightarrow{\mathcal{A}}^\delta_{K_1,K_2,C_0,C_1}$ of free orderings of $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ is Ramsey and has the expansion property.*

Proof. This follows by a combination of Theorem 2.1 and Lemma 4.18. The expansion property can be shown by the standard argument extending every edge by a fork with distances M and using the Ramsey property to force the ordering. □

We should give a citation for people to find this standard argument.

Theorem 4.20. *For every choice of admissible primitive parameters δ, K_1, K_2, C_0 and C_1, the class $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ has coherent EPPA.*

Proof. This follows by a combination of Theorem 2.2, Lemma 4.18 and Lemma 4.8. □

5. Primitive spaces with Henson contraints

In this section we extend results of Section 4 for classes with Henson constraints. An important fact about spaces with Henson constraints is the following [35, Section 1.3]:

Remark 5.1. Let O be a set of $(1,\delta)$-spaces. Then there exists a finite set $\mathcal{S} \subset O$, such that $\mathrm{Forb}(\mathcal{S}) = \mathrm{Forb}(O)$.

A reflexive and transitive relation on a set X is said to be a quasi order. It is a well quasi order[3] if every non empty subset has at least one but only finite many minimal elements. Now, up to isometry, $(1,\delta)$ spaces are characterised by the sizes of their cliques i.e., maximal subsets of vertices at mutual distance 1. If we express every such space as a finite non decreasing sequence of non negative integers we see that a $(1,\delta)$-space is embeddable in an other if and only if the latter contains a subsequence which majorises the former term by term. In order theory it is a well known fact that the set of finite sequences of a well quasi ordered set is a well quasi ordered set with respect to the above relation (see also [36] and [37]).

Lemma 5.1. *Let $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ be a primitive admissible sequence of parameters and let $\mathcal{S}$ be a non-empty class of Henson constraints. There is an M such that the completion with the magic parameter M does not introduce distances 1 or δ.*

Proof. For $\delta > 3$ the completion algorithm with magical parameter $M \geq \lceil \frac{\delta}{2} \rceil$ introduces distances δ or 1 either when $M = \delta$ or when 1 is the only possible completion of some fork.

For $K_1 < \delta$ we can choose M to be smaller than δ, hence the first case only appears if $K_1 = \delta$. However then by Definition 3.3, $\mathcal{S} = \emptyset$. The second case only appears if $C = 2\delta + 2$ and $C' = C + 1$, since then (δ, δ)-forks can only be completed by distance 1. Then again, by Definition 3.3, $\mathcal{S} = \emptyset$. □

Theorem 5.2. *Let $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ be a primitive admissible sequence of parameters and let $\mathcal{S}$ be a non-empty class of Henson constraints. For every magic parameter M we can define a* stationary independence relation with magic parameter *M on $\Gamma^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ as follows: $\mathbf{A} \mathop{\downarrow}_{\mathbf{C}} \mathbf{B}$ if and only if $\langle \mathbf{ABC} \rangle$ is isomorphic to the completion with magic parameter M of the free amalgamation of $\langle \mathbf{AC} \rangle$ and $\langle \mathbf{BC} \rangle$ over $\mathbf{C}$.*

Proof. This result can be shown as Corollary 4.17, the correctness of the completion algorithm can be verified by a combination of Theorem 4.9 and Lemma 5.1. □

Theorem 5.3. *Let $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ be a primitive admissible sequence of parameters such that $\mathcal{S}$ is non-empty class of Henson constraints. Then the class of $\overrightarrow{\mathcal{K}}$ of free linear orderings of $\mathcal{K} = \mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1} \cap \mathcal{A}_{\mathcal{S}}$ is Ramsey.*

Proof. Recall the definition of locally finite subclasses in Definition 2.6. We show that $\overrightarrow{\mathcal{K}}$ is a locally finite subclass of $\overrightarrow{\mathcal{A}}^{\delta}_{K_1,K_2,C_0,C_1}$. Given $\mathbf{C}_0 \in \overrightarrow{\mathcal{A}}^{\delta}_{K_1,K_2,C_0,C_1}$ we put $n = |C_0|$. Consider any $\mathbf{C}$ with a homomorphism to $\mathbf{C}_0$ such that every substructure with at most n vertices can be completed to $\mathbf{C}' \in \overrightarrow{\mathcal{K}}$. Because Henson constrains are complete edge-labelled graphs, we know that every Henson constraint in $\mathbf{C}$ is mapped to an isomorphic copy of itself by any homomorphism $\mathbf{C} \to \mathbf{C}_0$. Therefore there there is no $\mathbf{H} \in \mathcal{S}$, $|H| \geq n$ such that $\mathbf{H} \to \mathbf{C}$. Because every subgraph with at most n vertices can be completed to $\overrightarrow{\mathcal{K}}$ we also know that there is no $\mathbf{H} \in \mathcal{S}$, $|H| \leq n$ such that $\mathbf{H} \to \mathbf{C}$.

By Lemma 5.1 we know that the magic completion $\mathbf{C}'$ of $\mathbf{C}$ (which exists by Theorem 4.9) will not introduce any edges of distance 1 and δ and thus also $\mathbf{C}'$ contains no forbidden Henson constraints. The Ramsey property follows by Theorem 2.1. □

"magic completion" is too colloquial. Reword this. -Mike

[3]One of the authors opposes this name. *"Well,"* says he with pedantry, *"is an adverb, not an adjective."*

Theorem 5.4. *Let $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ be a primitive admissible sequence of parameters such that S is a non-empty class of Henson constraints. Then $\mathcal{K} = \mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1} \cap \mathcal{A}_{\mathcal{S}}$ has coherent EPPA.*

Proof. By Lemma 4.18 there is a finite set of obstacles O for $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$. By Remark 5.1 $\mathcal{S}$ is finite. Given $A \in \mathcal{K}$, apply Theorem 2.2 to obtain an extension $\mathbf{B} \in \mathrm{Forb}(O \cup \mathcal{S})$. Denote by $\mathbf{C}$ its completion with magic parameter M. By Lemma 5.1 we know that $\mathbf{C} \in \mathrm{Forb}(\mathcal{S})$ and by Lemma 4.9 we know that the automorphism group is unaffected. It follows that $\mathbf{C}$ is the desired completion. □

6. Bipartite 3-constrained spaces

In this section we discuss the bipartite classes of finite diameter in Cherlin's classification (Case (I) in Theorem 3.2), so we study classes of metric spaces $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$ with parameters

$$\delta < \infty, K_1 = \infty, K_2 = 0, C_1 = 2\delta + 1.$$

Furthermore we assume that

$$C_0 > 2\delta + 3.$$

The antipodal case where $C_0 = 2\delta + 2$ will be treated in Section 7. The parameter C_0 has to be even, so $2\delta + 3$ is not an acceptable value for C_0. We also discuss the Henson constraints for bipartite graphs.

By the condition $K_1 = \infty$, the metric spaces in $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$ contain no triangles of odd perimeter. As a direct consequence the relation consisting of all pairs (x, y) such that $d(x, y)$ is even, is an equivalence relation with two equivalence classes; this fact also motivates the name "bipartite 3-constrained spaces".

6.1. *Generalised completion algorithm for bipartite 3-constrained classes*

Our aim in this section is to again describe a procedure that completes a given edge-labelled graph $\mathbf{G}$ to metric spaces in $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$ whenever possible. The completion algorithm constructed in Section 4.1 fails in general in the bipartite setting, since it might introduce triangles with odd perimeter (for instance when adding magic distances in the final step). Hence Theorem 4.9 cannot be applied here.

In the following we show how we can slightly adapt the algorithm to ensure that no new odd cycles are generated. The basic idea for our completion algorithm is again to optimize the length of the newly introduced edges towards a magic parameter M, respectively its successor $M + 1$. The length of the remaining edges are then set to M or $M + 1$ depending on the parity prescribed by the bipartition.

Definition 6.1 (Bipartite magic distances)**.** Let $M \in \{1, 2, \dots, \delta\}$ be a distance. We say that M is *magic (with respect to $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$)* if

$$\left\lfloor \frac{\delta}{2} \right\rfloor \leq M < M + 1 \leq \left\lfloor \frac{C_0 - \delta - 1}{2} \right\rfloor.$$

Compare this with Definition 4.3. By the assumption $C_0 > 2\delta + 3$ a magic distance always exists.

Observation 6.1. *M is magic if and only if it has the following property: for all even $1 < b \leq \delta$ the triangles MMb and $(M + 1)(M + 1)b$ are in $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$; and for all odd $1 \leq b \leq \delta$ the triangle $M(M + 1)b$ is in $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$.*

Proof. If M has this property then $M \geq \left\lfloor \frac{\delta}{2} \right\rfloor$ (otherwise for even δ the triangle $MM\delta$ would be non-metric; similarly for odd δ the triangle $M(M + 1)\delta$ would be non-metric). Also, $M \leq \left\lfloor \frac{C_0-\delta-1}{2} \right\rfloor$ (otherwise if δ is even the triangle $(M + 1)(M + 1)\delta$ has perimeter C_0, hence is forbidden by the C_0 bound; and if δ is odd the triangle $M(M+1)\delta$ has perimeter C_0). The other implication follows from the definition of $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$. □

The following simple lemma will allow us to discuss completions only for *connected* edge-labelled graphs, i.e. edge-labelled graphs that there exists a path connecting each pair of vertices.

Lemma 6.2. $\mathbf{G} \in \mathcal{G}^{\delta}$ *has a completion to $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$ if and only if all of its connected components have a completion to $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$.*

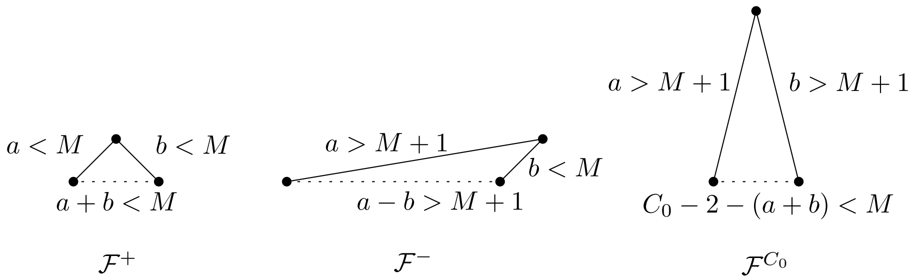

Figure 8: Forks used by $\mathbb{F}$ by the bipartite algorithm.

See Remark 6.1 for a discussion about the disconnected case.

Proof. It suffices to show that every $\mathbf{G} = (G, d)$ that is the disjoint union of two graphs A, B from $\mathcal{A}^\delta_{\infty,0,C_0,2\delta+1}$ has a completion $(G, d') \in \mathcal{A}^\delta_{\infty,0,C_0,2\delta+1}$.

Take $x \in A$ and $y \in B$. Then, for every non-edge (x', y') with $x' \in A$ and $y' \in B$ let $d'(x', y') = M$ if $d(x, x') + d(y, y')$ is even and $d'(x', y') = M + 1$ otherwise. It is not hard to verify that all the newly introduced triangles are of the form MMb, $(M + 1)(M + 1)b$ where b is even, or $M(M + 1)b$ where b is odd. Hence $(G, d') \in \mathcal{A}^\delta_{\infty,0,C_0,2\delta+1}$. □

For connected graphs we now give the following definition of a completion algorithm:

Definition 6.2 (Bipartite completion algorithm)**.** Given $1 \le M \le M + 1 \le \delta$, a one-to-one function $t : \{1, 2, \dots, \delta\} \setminus \{M, M + 1\} \to \mathbb{N}$ and a function $\mathbb{F}$ from $\{1, 2, \dots, \delta\} \setminus \{M, M+1\}$ to the power set of $\mathcal{D}$, then we define the $(\mathbb{F}, t, M, M + 1)$-*completion* of a connected edge-labelled graph $\mathbf{G} = (G, d)$ as the limit of the sequence $\mathbf{G}_1, \mathbf{G}_2, \dots$ that is constructed as in Definition 4.2; the length of all remaining non-edges (u, v) in this limit is then set to M if $d^+(u, v)$ has the same parity as M and to $M + 1$ otherwise. In addition we will stick to the other notational conventions introduced in Section 4.1 (time function, a expands to b,...).

Let M be a magic distance and let $1 \le x \le \delta$ with $x \neq M, M + 1$. Then we define the fork sets $\mathcal{F}^+_x$ and $\mathcal{F}^-_x$ as in the last section and $\mathcal{F}^{C_0}_x = \{(a, b) \in \mathcal{D} : C_0 - 2 - a - b = x\}$, i.e. $(a, b) \in \mathcal{F}^{C_0}_x$ if and only if $a + b + x$ is equal to $C_0 - 2$ and hence also even. Forks used by the bipartite algorithm are schematically depicted at Figure 8.

We further denote

$$\mathbb{F}_M(x) = \begin{cases} \mathcal{F}^+_x \cup \mathcal{F}^{C_0}_x & x < M \\ \mathcal{F}^-_x & x > M + 1. \end{cases}$$

For a magic distance M, we define the function $t_M : \{1, \dots, \delta\} \setminus \{M, M + 1\} \to \mathbb{N}$ as

$$t_M(x) = \begin{cases} 2x - 1 & x < M \\ 2(\delta - x) & x > M + 1. \end{cases}$$

We then call the $(\mathbb{F}_M, t_M, M, M+1)$-completion of $\mathbf{G}$ the *bipartite completion of* $\mathbf{G}$ *with magic parameter* M.

Lemma 6.3 (Bipartite Time Consistency Lemma)**.** *Let a, b be distances different from M and $M + 1$. If a expands to b, then $t_M(a) > t_M(b)$.*

Proof. Analogously to Lemma 4.3 we consider three types of forks.

$\mathcal{F}^+$: This follows in complete analogy to Lemma 4.3: If $a < M$ and $\mathcal{F}^+_a \neq \emptyset$, then $b < a < M$, hence $t_M(b) < t_M(a)$.

$\mathcal{F}^{C_0}$: If $a < M$ and $\mathcal{F}^{C_0}_a \neq \emptyset$, then we must have $b, c > M + 1$. Otherwise, if for instance $b \le M + 1$, then $C_0 - \delta - 2 \le C_0 - 2 - c = a + b < 2M + 1 \le 2\left\lfloor \frac{C_0-\delta-1}{2} \right\rfloor - 2 + 1$, a contradiction. As $C_0 \ge 2\delta + 4$, we obtain the inequality $b = (C_0 - 2) - c - a \ge (2\delta + 2) - \delta - a = \delta + 2 - a$. Hence $t_M(b) \le 2(a - 2) < 2a - 1 = t_M(a)$.

$\mathcal{F}^-$**:** Finally, we consider the case where $a > M + 1$ and $\mathcal{F}_a^- \neq \emptyset$. Then either $a = b - c$, which implies $b > a > M + 1$ and thus $t_M(b) < t_M(a)$, or $a = c - b$, which means $b = c - a \leq \delta - a$. Because of $a > M + 1 \geq \left\lceil \frac{\delta}{2} \right\rceil$, we have $b < M$. So $t_M(b) \leq 2(\delta - a) - 1 < 2(\delta - a) = t_M(a)$.

□

Lemma 6.4 (Bipartite $\mathbb{F}_M$ Completeness Lemma)**.** *Let* $\mathbf{G} \in \mathcal{G}^\delta$ *and* $\overline{\mathbf{G}}$ *be its completion with magic parameter M. If there is a triangle of even perimeter in* $\overline{\mathbf{G}}$ *that is either non-metric or forbidden due to the* C_0*-bound, and that contains an edge of length M or $M + 1$, then that edge was already in* $\mathbf{G}$.

Proof. By Observation 6.1 no triangles of type aMM, $aM(M + 1)$ or $a(M + 1)(M + 1)$ of even perimeter are forbidden. Suppose then that there is a forbidden triangle abN of even perimeter in $\overline{\mathbf{G}}$ such that $M \leq N \leq M + 1$ and the edge of length N is not in $\mathbf{G}$. For convenience define $t_M(M) = t_M(M + 1) = \infty$, which corresponds to the fact that edges of lengths M and $M + 1$ are added in the last step.

non-metric: If abN is non-metric then either $a + b < N$ or $|a - b| > N$. By Lemma 6.3 we have in both cases that $t_M(a + b)$ (respectively, $t_M(|a - b|)$) is greater than both $t_M(a)$ and $t_M(b)$. Therefore the completion algorithm would chose $a + b$ (resp. $|a - b|$) as the length of the edge instead of N. Observe that because abN has even perimeter it follows that this value differs from N by at least 2, so it is not equal to M or $M + 1$.

C_0**-bound:** If $a + b + N \geq C_0$ (which includes all the triangles forbidden by C_0-bound), then $t_M(C_0 - 2 - a - b) > t_M(a), t_M(b)$ by Lemma 6.3, so the algorithm would set $C_0 - 2 - a - b$ instead of N as the length of the third edge.

□

Lemma 6.5 (Bipartite Optimality and Parity Lemma)**.** *Let* $\mathbf{G} = (G, d) \in \mathcal{G}^\delta$ *be connected such that there is a completion of* $\mathbf{G}$ *into* $\mathcal{A}^\delta_{\infty,0,C_0,2\delta+1}$. *Denote by* $\overline{\mathbf{G}} = (G, \bar{d})$ *the completion of* $\mathbf{G}$ *with a magic parameter M and let* $\mathbf{G}' = (G, d') \in \mathcal{A}^\delta_{\infty,0,C_0,2\delta+1}$ *be an arbitrary completion of* $\mathbf{G}$. *Then for every pair of vertices $u, v \in G$ one of the following holds:*

1. $d'(u, v) \geq \bar{d}(u, v) \geq M + 1$,
2. $d'(u, v) \leq \bar{d}(u, v) \leq M$,
3. $M \leq \bar{d}(u, v) \leq M + 1$.

Furthermore the parities of $d'(u, v)$ and $\bar{d}(u, v)$ are equal.

Proof. The first part of the Lemma can be proven analogously to Lemma 4.6:

Suppose that the statement is not true and take any witness $\mathbf{G}' = (G, d')$ (i.e. a completion of $\mathbf{G}$ into $\mathcal{A}^\delta_{\infty,0,C_0,2\delta+1}$ such that there is a pair of vertices violating the statement). Recall that the completion with a magic parameter M is defined as a limit of a sequence $\mathbf{G}_1, \mathbf{G}_2, \ldots$ of edge-labelled graphs such that $\mathbf{G}_1 = \mathbf{G}$ and each two subsequent graphs differ at most by adding edges of a single distance.

Take the smallest i such that in the graph $\mathbf{G}_i = (G, d_i)$ there are vertices $u, v \in G$ with $d_i(u, v) > M + 1$ and $d_i(u, v) > d'(u, v)$ or $d_i(u, v) < M$ and $d_i(u, v) < d'(u, v)$. Let $w \in G$ be the witness of $d_i(u, v)$.

Now again we shall distinguish three cases, based on whether $d_i(u, v)$ was introduced by $\mathcal{F}^-$, $\mathcal{F}^+$ or $\mathcal{F}^{C_0}$. The cases $\mathcal{F}^-$ and $\mathcal{F}^+$ follow exactly the same way as in the proof of Lemma 4.6. We verify the remaining case $\mathcal{F}^{C_0}$.

Recall that, by the admissibility of C_0, we have $C_0 - 1 \geq 2\delta + 3$ and $M + 1 \leq \left\lfloor \frac{C_0 - \delta - 1}{2} \right\rfloor$. Thus we get $d_{i-1}(u, w), d_{i-1}(v, w) > M + 1$ (otherwise, if, say, $d_{i-1}(u, w) \leq M + 1$, we obtain contradiction $C_0 - \delta - 1 \geq 2M + 2 > d_{i-1}(u, w) + d_i(u, v) = C_0 - 2 - d_{i-1}(v, w) \geq C_0 - \delta - 2$ where $2M + 2 \neq C_0 - \delta - 2$ are both even but within an interval of size 1). So again by the Bipartite Optimality and the Parity Lemma 6.5 $d'(u, w) \geq d_{i-1}(u, w)$ and $d'(v, w) \geq d_{i-1}(v, w)$, which means that the triangle u, v, w in $\mathbf{G}'$ is forbidden by the C_0 bound, which is absurd as $\mathbf{G}'$ is a completion of $\mathbf{G}$ in $\mathcal{A}^\delta_{\infty,0,C_0,2\delta+1}$.

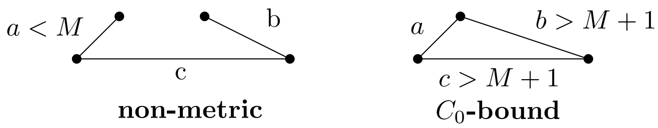

Figure 9: Types of forbidden triangles in bipartite spaces.

For the second part (about parities), observe first that the parity of an edge $d'(u, v)$ in $\mathbf{G}'$ has to be equal to $d'(u, w) + d'(w, v)$ for every other vertex w, since there are no triangles of odd perimeter in $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$. Since $\mathbf{G}$ is connected, the parity of $d'(u, v)$ is equal to the parity of the path distance of (u, v) in $\mathbf{G}$.

By the definition of the bipartite completion algorithm as a limit of graphs $\mathbf{G}_1, \mathbf{G}_2, \ldots$, if the statement is not true, then there has to be a smallest i such that in the graph $\mathbf{G}_i = (G, d_i)$ there are vertices $u, v \in G$ where the parity of $d_i(u, v)$ differs from the parity of $d'(u, v)$. Let w be a witness for the edge (u, v). Then three cases can appear (corresponding to $\mathcal{F}^-, \mathcal{F}^+, \mathcal{F}^{C_0}$).

$\mathcal{F}^-$: In the first case we have $d_i(u, v) = |d_{i-1}(u, w) - d_{i-1}(v, w)|$, which has the same parity as $d_{i-1}(u, w) + d_{i-1}(v, w)$. By minimality of i this value has the same parity as $d'(u, w) + d'(v, w)$, hence this case cannot appear.

$\mathcal{F}^+$: In the second case we have $d_i(u, v) = d_{i-1}(u, w) + d_{i-1}(v, w)$. Analogously to the first case then $d_i(u, v)$ has to have the same parity as $d'(u, w) + d'(v, w)$, which is a contradiction.

$\mathcal{F}^{C_0}$: In the third case $d_i(u, v) = C_0 - 2 - d_{i-1}(u, w) - d_{i-1}(v, w)$. Since $C_0 - 2$ is even, this distance has again the same parity $d_{i-1}(u, w) + d_{i-1}(v, w)$ and hence $d'(u, w) + d'(v, w)$, which is a contradiction.

In the last step, the distances M and $M + 1$ are added according to the parity of the path distance, hence also in this step the parity of edges is preserved. □

Note that Lemma 6.3 implies that any magic completion of a graph $\mathbf{G}$ contains a triangle with odd perimeter if and only if every completion of $\mathbf{G}$ contains such a triangle. In order to verify the correctness of our completion algorithm it is therefore only left to verify the analogous statement for the two other types of forbidden triangles: non-metric triangles, and even triangles that are forbidden due to the C_0-bound. But for those triangles the result can be shown just by following the corresponding proofs in the primitive case.

Lemma 6.6 (Bipartite Metric Lemma). *Let $\mathbf{G} = (G, d) \in \mathcal{G}^{\delta}$ be a connected edge-labelled graph such that there is a completion of $\mathbf{G}$ into $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$; let $\overline{\mathbf{G}} = (G, \bar{d})$ be its bipartite completion with magic parameter M. Then there are no non-metric triangles in $\overline{\mathbf{G}}$.*

Proof. We proceed in analogy to the proof of Lemma 4.11.

Suppose for a contradiction that there is a triangle with vertices $u, v, w \in \overline{\mathbf{G}}$ such that $\bar{d}(u, v) + \bar{d}(v, w) < \bar{d}(u, w)$. Denote $a = \bar{d}(u, v)$, $b = \bar{d}(v, w)$ and $c = \bar{d}(u, w)$ and assume without loss of generality that $a \le b < c$. By Lemma 6.5 we know that $a + b + c$ is even. Let a', b', c' be the corresponding edge lengths in an arbitrary completion of $\mathbf{G}$ into $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$. We shall distinguish three cases:

1. First suppose $a, b, c < M$. Then $t_M(a) \le t_M(b) < t_M(a + b) < t_M(c)$, which means that c must be already in $\mathbf{G}$. By Lemma 6.5 $a' + b' \le a + b < c = c'$, which is a contradiction.

2. Another possibility is $a < M$ and $b, c \ge M$ (actually $c > M + 1$, since abc is non-metric). By Lemma 6.5 we know that $a' \le a$ and $c' \ge c$. If b was already in $\mathbf{G}$, then $\mathbf{G}$ has no completion – a contradiction. Otherwise clearly $c-a > b \ge M$, so $t_M(c-a) < t_M(b)$ (define $t_M(M) = t_M(M+1) = \infty$). But as $c - a$ expands to c and a, we get $t_M(c - a) > t_M(c), t_M(a)$, which means that the bipartite completion algorithm with magic parameter M would complete the edge v, w with the length $c - a$ and not with b.

3. The last possibility is $a, b < M$ and $c \ge M$. Then (by Lemma 6.5 and Lemma 6.4 if $M \le c \le M + 1$) we have $a' \le a$, $b' \le b$ and $c' \ge c$, hence the triangle a', b', c' is again non-metric.

□

Lemma 6.7 (Bipartite C_0-bound Lemma)**.** *Let $\overline{\mathbf{G}}, \mathbf{G}$ be as in Lemma 6.6. Then there are no triangles forbidden by the C_0-bound in $\overline{\mathbf{G}}$.*

Proof. Again we proceed analogously to Lemma 4.10. Suppose for contradiction that there is a triangle with vertices u, v, w in $\overline{\mathbf{G}}$ such that $\bar{d}(u, v) + \bar{d}(v, w) + \bar{d}(u, w) \geq C_0$. For brevity let $a = \bar{d}(u, v)$, $b = \bar{d}(v, w)$ and $c = \bar{d}(u, w)$. Assume without loss of generality $a \leq b \leq c$. Let a', b', c' be the corresponding edge lengths in an arbitrary completion of $\overline{\mathbf{G}}$ into $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$. Then two cases can appear.

Either $a, b, c > M + 1$, and then by Lemma 6.5 we have $a' \geq a$, $b' \geq b$ and $c' \geq c$, so we get the contradiction $a' + b' + c' \geq C_0$; or $a \leq M + 1$, $c \geq b > M + 1$ and $a + b + c \geq C_0$. In this case Lemmas 6.4 and 6.5 imply $b' \geq b$ and $c' \geq c$ and $a' \leq a$. If the edge (u, v) was already in $\mathbf{G}$, then clearly $a' + b' + c' \geq a + b + c \geq C_0$, which is a contradiction. If (u, v) was not already an edge in $\mathbf{G}$, then it was added by the bipartite completion algorithm with magic parameter M in step $t_M(a)$. Let $\bar{a} = C_0 - 2 - b - c$. Then clearly $\bar{a} < a$, which means that $t_M(\bar{a}) < t_M(a)$, and as $\bar{a}$ expands to b, c, we have $t_M(b), t_M(c) < t_M(\bar{a})$. But then the bipartite completion with magic parameter M actually sets the length of the edge u, v to be $\bar{a}$ in step $t_M(\bar{a})$, which is a contradiction. □

Lemma 6.8 (Bipartite automorphism Preservation Lemma)**.** *Let $\mathbf{G} \in \mathcal{G}^{\delta}$ be connected and $\overline{\mathbf{G}}$ be its bipartite completion with a magic parameter M. Then every automorphism of $\mathbf{G}$ is also an automorphism of $\overline{\mathbf{G}}$.*

Proof. Cf. proof of Lemma 4.8. Observe that since $\mathbf{G}$ is assumed to be connected, the final step of the algorithm that includes edges of length M and $M+1$ is canonical. Hence automorphisms are preserved. □

Theorem 6.9. *Let $\delta \geq 3$, $C_0 > 2\delta+3$ and $\mathcal{S}$ be an admissible set of Henson constraints for $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$. Let $\mathbf{G} = (G, d)$ be a connected edge-labelled graph such that there is a completion of $\mathbf{G}$ into $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1} \cap \mathcal{A}_{\mathcal{S}}$ and let $\overline{\mathbf{G}} = (G, \bar{d})$ be its bipartite completion with magic parameter M. Then $\overline{\mathbf{G}} \in \mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1} \cap \mathcal{A}_{\mathcal{S}}$.*

$\overline{\mathbf{G}}$ is optimal in the following sense: Let $\mathbf{G}' = (G, d') \in \mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$ be an arbitrary completion of $\mathbf{G}$ in $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1,\mathcal{S}}$, then for every pair of vertices $u, v \in G$ one of the following holds:

1. *$d'(u, v) \geq \bar{d}(u, v) \geq M + 1$,*
2. *$d'(u, v) \leq \bar{d}(u, v) \leq M$,*
3. *$M \leq \bar{d}(u, v) \leq M + 1$.*

Furthermore the parity of every distance in $\mathbf{G}'$ is the same as the parity of the corresponding distance in $\overline{\mathbf{G}}$ and every automorphism of $\mathbf{G}$ is also an automorphism of $\overline{\mathbf{G}}$.

Proof. For $\mathcal{S} = \emptyset$ the statement follows from Lemmas 6.6, 6.7, 6.5 and 6.8.

Observe that the only non-empty admissible set $\mathcal{S}$ consists of a single anti-clique (that is a metric space with all distances δ) and in this case δ is even. Then we can use the fact that $\delta \geq 4$ and thus M can be always chosen to be at most $\delta - 2$. In this case the bipartite completion with magic parameter M will never introduce edge of distance δ. □

Remark 6.1. Note that if we drop the condition of $\mathbf{G}$ being connected in Theorem 6.9, we can still compute a completion $\overline{\mathbf{G}}$ of $\mathbf{G}$ by first completing all connected components according to Theorem 6.9 and then adding edges M and $M + 1$ as described in the proof of Lemma 6.2. This completion $\overline{\mathbf{G}}$ still satisfies the optimality conditions 1,2 and 3; however we will lose other important features:

1. The constructed completion is not uniquely determined by $\mathbf{G}$, but also depends on how we connect the different components of $\mathbf{G}$. (The proof of Lemma 6.2 used a non canonical choice of $x \in A$ and $y \in B$, two connected components.)
2. The automorphism group of $\overline{\mathbf{G}}$ can be a proper subgroup of $\mathrm{Aut}(\mathbf{G})$,
3. Edges in $\mathbf{G}'$ and $\overline{\mathbf{G}}$ may have different parities.

The above observations have an impact on the results of the following section: Point 1 corresponds to the fact that we only have local, but not global stationary independence relation (see Corollary 6.10).

6.2. Local stationary independence relation

Corollary 6.10. *Let $\delta \geq 3$, $C_0 > 2\delta + 3$ and $\mathcal{S}$ be a non-empty, admissible set of Henson constraints for $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$. Then there is no stationary independence relation on $\Gamma^{\delta}_{\infty,0,C_0,2\delta+1,\mathcal{S}}$. However, for every magic parameter M there is a local stationary independence relation on $\Gamma^{\delta}_{\infty,0,C_0,2\delta+1,\mathcal{S}}$ as follows: $\mathbf{A} \mathop{\downarrow}_{\mathbf{C}} \mathbf{B}$ if and only if $\langle \mathbf{ABC} \rangle$ is isomorphic to the completion of the completion with a magic parameter M of the free amalgamation of $\langle \mathbf{AC} \rangle$ and $\langle \mathbf{BC} \rangle$ over $\mathbf{C}$.*

Proof. First, suppose for a contradiction that there is a stationary independence relation $\mathop{\downarrow}$ in $\Gamma^{\delta}_{\infty,0,C_0,2\delta+1}$. By Theorem 4.16 this is equivalent to the existence of a canonical amalgamation operator $\oplus$ on $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$. Let $\mathbf{A}, \mathbf{B} \in \mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$ such that $\mathbf{A}$ consist of two vertices u, v with $d(u, v) = 1$ and let $\mathbf{B}$ contains only one vertex w. In the canonical amalgam over the empty set $\mathbf{A} \oplus \mathbf{B}$, monotonicity implies that both $(\{u, w\}, d)$ and $(\{u, v\}, d)$ are canonical amalgams of two points over the empty set. By the uniqueness of $\oplus$ we have that $d(u, w) = d(v, w)$. But then the triangle (u, v, w) has odd perimeter, which contradicts to $\mathbf{A} \oplus \mathbf{B} \in \mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$.

The local stationary independence relation follows by same argument as in proof of Corollary 4.17. By locality of the stationary independence relation we conclude that all graphs to be completed in the proof are connected and thus Theorem 6.9 applies. □

6.3. Ramsey property and EPPA

We follow the general direction of Section 4.6. The extra difficulty is that the bipartiteness can not be expressed by means of a finite set of obstacles, because this finite set must contain odd cycles of unbounded length. We however can obtain the following variant of Lemma 4.18.

Lemma 6.11 (Bipartite Finite Obstacles Lemma)**.** *Let $\delta \geq 3$, $C_0 > 2\delta + 3$. Then there is finite set $\mathcal{O}$ of edge-labelled cycles such that every edge-labelled graph $\mathbf{G} \in \mathrm{Forb}(\mathcal{O})$ without odd cycles is in $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$*

Proof. Cf. proof of Lemma 4.18. To verify that the final step of algorithm will succeed we use the fact that there are no odd cycles in $\mathbf{G}$. □

Theorem 6.12. *Let $\delta \geq 3$, $C_0 > 2\delta + 3$ and $\mathcal{S}$ be an admissible set of Henson constraints for $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$. The class $\overrightarrow{\mathcal{A}}^{\delta}_{\infty,0,C_0,2\delta+1} \cap \overrightarrow{\mathcal{A}}_{\mathcal{S}}$ of convex orderings of $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1} \cap \mathcal{A}_{\mathcal{S}}$ with an additional unary predicate determining the bipartition is Ramsey and has the expansion property.*

Here a convex ordering is any ordering such that vertices in first bipartition (denoted by the unary predicate) form an initial segment.

MK: We should also cite here the Ramsey result on which this is based on.

Proof. Let $\mathcal{O}$ be given by Lemma 6.11. Denote by $\mathcal{O}'$ the family of all possible expansions of $\mathcal{O}$ by a unary predicate determining the bipartition with additional structure on two vertices which forbids vertices from the same bipartition from being connected by an edge of odd length and vertices from different bipartitions from being connected by an edge of even length. Observe that structures in $\mathrm{Forb}(\mathcal{O}')$ have no odd cycles and thus Theorem 2.1 and Lemma 6.11 apply.

If $\mathcal{S}$ is non-empty, then observe that M can be chosen to be at most $\delta - 2$ and use same argument as in proof of Theorem 5.3.

The expansion property again follows by the standard argument. □

Theorem 6.13. *Let $\delta \geq 3$, $C_0 > 2\delta + 3$ and $\mathcal{S}$ be an admissible set of Henson constraints for $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1}$. Then the class $\mathcal{A}^{\delta}_{\infty,0,C_0,2\delta+1} \cap \mathcal{A}_{\mathcal{S}}$ has coherent EPPA.*

Proof. Let $\mathcal{O}$ be given by Lemma 6.11. By an application of Theorem 2.2 obtain $\mathbf{B} \in \mathrm{Forb}(\mathcal{O})$ which is a coherent extension of $\mathbf{A}$. Without loss of generality we can assume that $\mathbf{B}$ is connected (otherwise the connected component of $\mathbf{B}$ containing $\mathbf{A}$ is a coherent extension, too). If $\mathbf{B}$ has no odd cycles, apply Theorem 6.9 to obtain the desired extension of $\mathbf{A}$. Note that $\mathbf{B}$ may contain odd cycles.

If $\mathbf{B}$ contains odd cycles, construct $\mathbf{C} = (C, d')$ as follows. The vertex set C is $B \times \{0, 1\}$, and

$$d'((u, i), (v, j)) = \begin{cases} 0 & \text{if } (u, i) = (v, j) \\ d(u, v) & \text{if } d(u, v) \text{ is odd and } i \neq j \\ d(u, v) & \text{if } d(v, j) \text{ is even and } i = j. \end{cases}$$

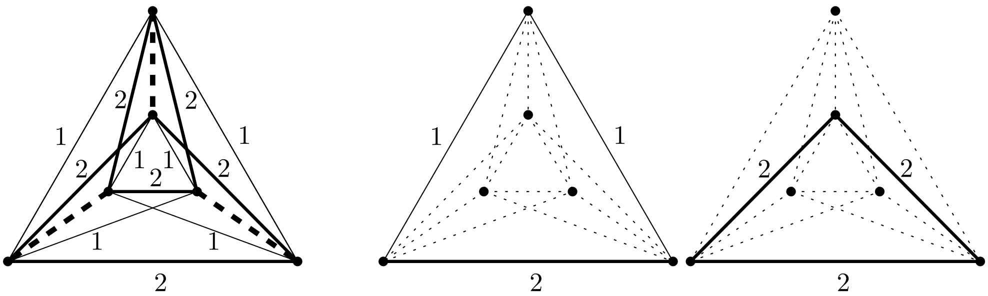

Figure 10: Antipodal extension of the triangle 122 for $\delta = 3$, the matching formed by edges of distance 3 is denoted by dashed lines. The second and third picture highlight the original triangle and one of its antipodal companions.

Observe that $\mathbf{C}$ is connected and contains no odd cycles. Let $p : A \to \{0,1\}$ be a function determining the bipartition of $\mathbf{A}$. We verify that $\mathbf{C}$ is an extension of $\phi(\mathbf{A})$ for the following embedding $\phi(v) = (v, p(v))$. Denote by $\pi(v, j) = v$ the projection, which is an homomorphism of $\mathbf{C} \to \mathbf{B}$ and also embedding $\phi(\mathbf{A}) \to \mathbf{A}$.

Every partial isometry ψ of $\phi(\mathbf{A})$ induces a partial isometry $\psi \circ \pi$ of $\mathbf{A}$ which extends to automorphism $\bar{\psi}$ of $\mathbf{B}$. This automorphism induces an automorphism ψ' of $\mathbf{C}$. It however might not be an extension of ψ because ψ may change the values of the function p. In this case combine it with an automorphism mapping $(v, 0) \mapsto (v, 1)$ and $(v, 1) \mapsto (v, 0)$. It is easy to see that this construction preserves coherency of the extensions in $\mathbf{B}$.

If $\mathcal{S}$ is non-empty, again observe that M can be chosen to be at most $\delta - 2$ and use same argument as in proof of Theorem 5.4. □

7. Antipodal spaces

In this section we discuss the antipodal classes in Cherlin's catalogue. We say that an amalgamation class $\mathcal{K}$ of metric spaces of diameter δ is *antipodal* if the edges of distance δ form a matching in the Fraïssé limit of $\mathcal{K}$. In particular then there are no triangles with more than one edge of length δ in $\mathcal{K}$. Note that this also implies that $\mathcal{K}$ has no strong amalgamation. For admissible parameters $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ the class $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ is antipodal if and only if $C = 2\delta + 1$. More precisely only the two cases can appear:

Definition 7.1. The admissible parameters $3 \leq \delta < \infty$, K_1, K_2, C_0 and C_1 are *antipodal* when

(I) $K_1 = \infty$, $C_0 = 2\delta + 2$ (the bipartite case; so $K_2 = 0$ and $C_1 = 2\delta + 1$), or

(IIA) $K_1 \leq \frac{\delta}{2}, C_0 = 2\delta + 2, C_1 = 2\delta + 1, K_2 = \delta - K_1$.

The parameters in this case are pushed to the extreme situation where, either there is no magic parameter, or $M = \lfloor \frac{\delta}{2} \rfloor$ is the only parameter satisfying Definition 4.4, respectively Definition 6.1.

Definition 7.2. Let $\delta \geq 3$. For $\mathbf{A} = (A, d) \in \mathcal{G}^{\delta-1}$, an *antipodal companion* of $\mathbf{A}$ is any $\mathbf{A}^* = (A, d^*) \in \mathcal{G}^{\delta-1}$ such that there exists $B \subseteq A$ and:

$$d^*(u, v) = \begin{cases} d(u, v) & \text{if } d(u, v) \text{ is defined and } u, v \in B \text{ or } u, v \notin B \\ \delta - d(u, v) & \text{if } d(u, v) \text{ is defined and } u \in B, v \notin B \text{ or vice versa.} \end{cases}$$

Observe that for every admissible antipodal parameters $\delta, K_1, K_2, C_0, C_1$, the class $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$, and also the subclass of metric spaces of diameter at most $\delta - 1$ form amalgamation classes. This subclass is equal to $\mathcal{A}^{\delta-1}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ and corresponds to either a primitive (Case (IIA) for $K_1 > 2$ or (III) for $1 \leq K_1 \leq 2$) or bipartite Case (I) of Cherlin's catalogue. The following describes a reverse way to produce an antipodal space of diameter δ from a space of diameter $\delta - 1$.

Definition 7.3 (Antipodal extensions)**.** Given an edge-labelled graph $\mathbf{M} = (M, d) \in \mathcal{G}^{\delta-1}$ its *antipodal extension* is the edge-labelled graph $\mathbf{M}^\ominus = (M \times \{0,1\}, d') \in \mathcal{G}^{\delta-1}$ such that $d'((u, i), (v, i)) = d(u, v)$ and $d'((u, i), (v, 1-i)) = \delta - d(u, v)$ for $u, v \in M$ and $i \in \{0, 1\}$.

For an ordered edge-labelled graph $\overrightarrow{\mathbf{M}} = (M, d, \leq_\mathbf{M})$, its *ordered antipodal extension* $\overrightarrow{\mathbf{M}}^\ominus = (M \times \{0,1\}, d', \leq_{\mathbf{M}^\ominus})$ where $(M, d)^\ominus = (M \times \{0,1\}, d')$ and $(u, i) \leq_{\mathbf{M}^\ominus} (v, j)$ if and only if $i < j$ or $i = j$ and $u \leq_\mathbf{M} v$.

7.1. *Ramsey property*

The Ramsey property follows from the above correspondence in full generality. Recall that we use arrows to indicate that the structures in a class are ordered (a requirement for any Ramsey class).

Theorem 7.1. *Let $\overrightarrow{\mathcal{K}}$ be a Ramsey class of expansions of metric spaces. Then the class $\overrightarrow{\mathcal{K}}^{\ominus}$ of all antipodal expansions of $\overrightarrow{\mathcal{K}}$ is Ramsey.*

Here $\overrightarrow{\mathcal{K}}$ can be either a Ramsey class of ordered metric spaces (given by Theorems 4.19 and 5.3) or ordered metric spaces with predicate denoting bipartition (given by Theorem 6.12).

MK: $\overrightarrow{\mathcal{K}}^{\ominus}$ is not defined for $\mathcal{K}$ containing unary predicates.

Proof. Given $\mathbf{A}^{\ominus}, \mathbf{B}^{\ominus} \in \overrightarrow{\mathcal{K}}^{\ominus}$, apply the Ramsey property in $\overrightarrow{\mathcal{K}}$ to obtain $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. It is easy to check that $\mathbf{C}^{\ominus} \longrightarrow (\mathbf{B}^{\ominus})^{\mathbf{A}^{\ominus}}_2$. □

Remark 7.1. Observe that $\overrightarrow{\mathcal{K}}^{\ominus}$ is not a hereditary class. It consists only of metric spaces where for every vertex there is precisely one vertex in a distance δ. Thus one can color only structures $\mathbf{A}$ having this property. The Ramsey property for antipodal expansions can be, equivalently, stated for the hereditary class of all subspaces of spaces in $\overrightarrow{\mathcal{K}}^{\ominus}$ with a unary predicate determining the podality.

This unary predicate becomes necessary because in $\overrightarrow{\mathcal{K}}^{\ominus}$ there is a definable equivalence relation on vertices which is not in $\mathcal{K}^{\ominus}$, namely $u \sim v$ whenever $\{w : w \leq u\}$ spans an edge of length δ iff $\{w : w \leq v\}$ does. As shown in [5] an equivalence relation on vertices implies the necessity for unary predicates in the Ramsey lift. It is interesting to observe that in showing EPPA (see Section 7.3) there is sometimes no need for further expansion of the language. To maintain that, it is necessary to assume that $\mathcal{K}$ is closed under the operation of forming antipodal companions (Definition 7.2).

7.2. *Generalised completion algorithms for antipodal classes*

To show the extension property for partial automorphisms we need a way to complete the spaces symmetrically. We start with an easy observation:

MK: Do we ever use Lemma 7.2?

Lemma 7.2. *Let $\mathcal{K}$ be a 3-constrained antipodal metric space of diameter δ from Cherlin's catalogue. Then the subclass $\mathcal{K}^{\delta-1}$ of all metric spaces in $\mathcal{K}$ with no edges of length δ is an amalgamation class closed under antipodal companions.*

Proof. It suffices to show that every antipodal companion of a triangle in $\mathcal{K}$ is in $\mathcal{K}$. Denote by $\mathbf{M}$ a triangle on vertices u, v, w containing no edges of length δ. By a sequence of 3 amalgamations with the metric space $\mathbf{E}$ consisting of two vertices at distance δ, we obtain $\mathbf{M}' \in \mathcal{K}$ with 6 vertices u, v, w, x, y, z. The triangle x, y, z is isometric to u, v, w and any antipodal companion of $\mathbf{M}$ is a substructure of $\mathbf{M}'$.

It follows that $\mathcal{K}^{\delta-1}$ is closed under antipodal companions. Given $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}^{\delta-1}$ the amalgamation of $\mathbf{B}_1^{\ominus}$ and $\mathbf{B}_2^{\ominus}$ over $\mathbf{A}^{\ominus}$ (in $\mathcal{K}$) contains also an amalgamation of $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2$ in $\mathcal{K}^{\delta-1}$. Therefore, $\mathcal{K}^{\delta-1}$ is also an amalgamation class. □

Our completion algorithm for the antipodal classes $\mathcal{K} = \mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ will be based on the completion algorithm for $\mathcal{K}^{\delta-1} = \mathcal{A}^{\delta-1}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ with magic parameter $M = \lfloor\frac{\delta}{2}\rfloor$. More precisely, we are not considering the completion to $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$, but to its expansion $\mathcal{B}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ by an additional unary predicate P determining the podality (so every edge of length δ has precisely one vertex $v \in P$). Roughly speaking our completion algorithm for an input structure (G, d, P) will first complete the pode (P, d) in $\mathcal{A}^{\delta-1}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ and then forms its antipodal extension.

We are going to consider four separate cases:

1. antipodal classes in Case (IIA) with even δ (Corollary 7.5),
2. antipodal bipartite classes in Case (I) with odd δ (Corollary 7.7),
3. antipodal classes in Case (IIA) with odd δ (Theorem 7.4), and
4. antipodal bipartite classes in Case (I) with even δ (Theorem 7.6).

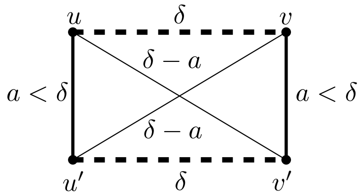

Figure 11: Antipodal quadruple.

In the first two cases we will show the perhaps surprising fact that the antipodal completion of (G, d, P) described as above does not depend on the choice of P, hence we obtain a unique completion $(G, \bar{d})$ to the original class of metric spaces $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$. In the latter two cases this however does not hold and similar to the bipartite case (cf. Remark 6.1) we are confronted with different completions, depending on the choice of P. This ambiguity cannot be eliminated and is reflected in the fact that there is no canonical amalgamation on $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ for those cases (Theorem 7.9).

In order to simplify our proof we first show that it is enough to only consider edge-labelled graphs $\mathbf{G}$ that are symmetric according to the following definition:

Definition 7.4 (Antipodal quadruple)**.** A quadruple of distinct vertices (u, v, u', v') in $\mathbf{G} \in \mathcal{G}^\delta$ is *antipodal* if $d(u, v) = d(u', v') = \delta$ and $d(u, u') = d(v, v')$, $d(u, v') = d(v, u') = \delta - d(u, u')$. (In particular all those distances are defined.) See Figure 11.

Let us call $\mathbf{G} \in \mathcal{G}^\delta$ *antipodally symmetric*, if for every vertex x there is a unique x^* with $d(x, x^*) = \delta$ and if every edge of length $< \delta$ is part of an antipodal quadruple.

Lemma 7.3. *Let $\mathbf{G} = (G, d) \in \mathcal{G}^\delta$ and suppose that $\mathbf{G}$ has a completion to $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$. Then there is an edge-labelled graph (G^*, d^*) with $G^* \supseteq G$ and $d^* \supseteq d$ that is antipodally symmetric and every completion of $(G, d') \in \mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ has a unique extension to a completion of (G^*, d^*). Furthermore every automorphism of (G, d) extends uniquely to an automorphism of (G^*, d^*).*

Proof. First observe that if $\mathbf{G}$ has a completion to $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ then there are no (δ, δ)-forks in $\mathbf{G}$. Moreover quadruples of vertices (u, v, u', v') with $d(u, v) = d(u', v') = \delta$ can only be completed to antipodal quadruple. Instead of completing (G, d) we can therefore consider the antipodally symmetric graph (G^*, d^*) that is constructed by adding a unique vertex x^* for every x and with $d^*(x, x^*) = \delta$ (if x^* is not already present in G) and completing all quadruples of vertices (u, v, u', v') such that $d^*(u, v) = d^*(u', v') = \delta$ and $d(u, u')$ is defined to their unique completion as antipodal quadruple. It is not hard to see that a completion $(G, d') \in \mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ of (G, d) gives rise to a unique completion of (G^*, d^*) into $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ by the setting $d^*(x^*, y) = \delta - d'(x, y)$ and $d^*(x^*, y^*) = d'(x, y)$, for $d(x, x^*) = d(y, y^*) = \delta$.Also every automorphism $f \in \mathrm{Aut}(G, d)$ extends to a unique automorphism $f \in \mathrm{Aut}(G^*, d^*)$ defined by $f(x^*) = f(x)^*$. □

Analogously, a structure (G, d, P) with $(G, d) \in \mathcal{G}^\delta$ and a unary predicate P has a completion to $(G, d', P) \in \mathcal{B}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ if and only if its extension to a antipodally symmetric space has a completion to $\mathcal{B}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$. Therefore, in the following we always assume (G, d, P) to be antipodally symmetric.

Note that, for antipodally symmetric (G, d, P), every completion of (P, d) to $\mathcal{A}^{\delta-1}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ extends to a unique completion of (G, d, P) to $\mathcal{B}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$, namely its antipodal expansion.

Definition 7.5 (Non-bipartite antipodal completion algorithm)**.** Let $\delta \geq 3$, $K_1 \leq \frac{\delta}{2}$, $M = \lfloor \frac{\delta}{2} \rfloor$, $C = 2\delta + 1$ and let (G, d, P) be an antipodally symmetric edge-labelled graph with a predicate P for a pode.

Then we define the *antipodal completion* $(G, \bar{d}, P)$ *of* (G, d, P) *with parameter* M as follows: restricted to the pode, $(P, \bar{d})$ is the completion of (P, d) with magic parameter M and diameter $\delta - 1$ (cf. Definition 4.4) and $(G, \bar{d})$ is its antipodal expansion. Since (G, d, P) is antipodally symmetric, $(G, \bar{d}, P)$ is in fact a completion of (G, d, P), i.e. $d(x, y) = \bar{d}(x, y)$, whenever $d(x, y)$ is defined.

Then the following holds:

Theorem 7.4. *Let $3 \leq \delta < \infty$ and $K_1 \leq \frac{\delta}{2}$ and $M = \lfloor \frac{\delta}{2} \rfloor$. Let (G, d, P) be antipodally symmetric. Suppose that (G, d, P) has a completion into $\mathcal{B}^{\delta}_{K_1,\delta-K_1,2\delta+2,2\delta+1,\mathcal{S}}$ and let $(G, \bar{d}, P)$ be its antipodal completion with magic parameter M. Then $(G, \bar{d}, P) \in \mathcal{B}^{\delta}_{K_1,\delta-K_1,2\delta+2,2\delta+1,\mathcal{S}}$ and it is optimal in the following sense: Let $(G, d', P) \in \mathcal{B}^{\delta}_{K_1,\delta-K_1,2\delta+2,2\delta+1,\mathcal{S}}$ be an arbitrary completion of (G, d, P), then, for all $u, v \in G$:*

1. *$d'(u, v) \geq \bar{d}(u, v) \geq M$ or $d'(u, v) \leq \bar{d}(u, v) \leq M$ if both $u, v \in P$ or if both $u, v \in G \setminus P$*
2. *$d'(u, v) \geq \bar{d}(u, v) \geq \delta - M$ or $d'(u, v) \leq \bar{d}(u, v) \leq \delta - M$ else.*

Furthermore $\mathrm{Aut}(G, d, P) = \mathrm{Aut}(G, \bar{d}, P)$.

Proof. An antipodally symmetric graphs (G, d, P) has a completion to $\mathcal{B}^{\delta}_{K_1,\delta-K_1,2\delta+2,2\delta+1,\mathcal{S}}$ if and only if its pode (P, d) has a completion to $\mathcal{A}^{\delta-1}_{K_1,\delta-K_1,2\delta+2,2\delta+1,\mathcal{S}}$.

By Theorem 4.9 the optimality statement $d'(u, v) \geq \bar{d}(u, v) \geq M$ or $d'(u, v) \leq \bar{d}(u, v) \leq M$ is true for all $u, v \in P$. In general, every pair of vertices u, v in G is part of an antipodal quadruple. This implies the optimality statement in its general form.

The fact that $\mathrm{Aut}(G, d, P) = \mathrm{Aut}(G, \bar{d}, P)$ follows directly from the automorphism preservation Lemma 4.8 for the pode (P, d) and the observation that every automorphism of (G, d, P) is uniquely determined by its restriction to P. $\square$

Corollary 7.5. *Let $3 \leq \delta < \infty$ be even, $K_1 \leq \frac{\delta}{2}$ and $M = \frac{\delta}{2}$. Let (G, d, P) be antipodally symmetric. Suppose that $\mathbf{G} = (G, d)$ has a completion into $\mathcal{A}^{\delta}_{K_1,\delta-K_1,2\delta+2,2\delta+1,\mathcal{S}}$. Then there is a unique completion $(G, \bar{d}) \in \mathcal{A}^{\delta}_{K_1,\delta-K_1,2\delta+2,2\delta+1,\mathcal{S}}$ that is optimal in the following sense: Let $\mathbf{G}' = (G, d') \in \mathcal{A}^{\delta}_{K_1,\delta-K_1,2\delta+2,2\delta+1,\mathcal{S}}$ be an arbitrary completion of $\mathbf{G}$, then, for all $u, v \in G$:*

$$d'(u, v) \geq \bar{d}(u, v) \geq M \text{ or } d'(u, v) \leq \bar{d}(u, v) \leq M.$$

Furthermore $\mathrm{Aut}(G, d) = \mathrm{Aut}(G, \bar{d})$.

Proof. We define $(G, \bar{d})$ as the underlying metric space of the completion of (G, d, P) for an arbitrary predicate P for a pode, i.e. for some P such that $|\{x, x^*\} \cap P| = 1$ for all x, x^* with $d(x, x^*) = \delta$. By Theorem 7.4, $(G, \bar{d}) \in \mathcal{A}^{\delta}_{K_1,\delta-K_1,2\delta+2,2\delta+1,\mathcal{S}}$, if and only if (G, d) has a completion to $\mathcal{A}^{\delta}_{K_1,\delta-K_1,2\delta+2,2\delta+1,\mathcal{S}}$.

Since δ is even, we have that $M = \delta - M$. Hence, the optimality part of Theorem 7.4 states that actually for all $u, v \in G$ and every arbitrary completion $\mathbf{G}' = (G, d') \in \mathcal{A}^{\delta}_{K_1,\delta-K_1,2\delta+2,2\delta+1,\mathcal{S}}$:

$$d'(u, v) \geq \bar{d}(u, v) \geq M \text{ or } d'(u, v) \leq \bar{d}(u, v) \leq M.$$

Hence $(G, \bar{d})$ does not depend on the choice of the pode P.

It remains to show that automorphisms of (G, d) (that do not necessarily fix a pode P) are preserved by the completion. So let $f \in \mathrm{Aut}(G, d)$ and P some predicate for a pode. Then (G, d, P) and $(G, d, f(P))$ are isomorphic under f. Hence also their completions are isomorphic, thus $f \in \mathrm{Aut}(G, \bar{d})$. $\square$

With the following example we would like to illustrate out that the assumption of δ to be even cannot be dropped in Corollary 7.5 and also the original completion algorithm fails for odd δ.

Example. Consider $\mathcal{A}^{3}_{0,10,7,2}$ which is an example of a class of antipodal metric spaces. Completing a graph consisting of two edges of length 3 and no other edges using Definition 4.4 will result in a non-antipodal metric space where every non-edge will be completed by $M = 2$. Using the completion in Defintion 7.6 for some choice of P we obtain a completion, which depends on the choice of the pode P and that has only the identity as automorphism, in contrast to the input graph.

Next we are going to consider the bipartite cases. Again Lemma 7.3 allows us to only consider antipodally symmetric edge-labelled graphs. Then we define the bipartite antipodal completion of (G, d, P) as follows:

Definition 7.6 (Bipartite antipodal completion algorithm)**.** Given $\delta \geq 3$, $M = \lfloor \frac{\delta}{2} \rfloor$ and $C = 2\delta + 1$ and an antipodally symmetric, connected edge-labelled graph (G, d, P) with a predicate P for a pode.

Then we define the *antipodal completion* $(G, \bar{d}, P)$ *of* (G, d, P) *with parameter* M by setting $(P, \bar{d})$ to be the bipartite completion of (P, d) with magic parameter M and diameter $\delta - 1$ (according to Definition 6.2) and $(G, \bar{d})$ to be its antipodal expansion. Since (G, d, P) is antipodally symmetric, $(G, \bar{d}, P)$ is in fact a completion of (G, d, P).

Theorem 7.6. *Let $3 \leq \delta$ and $M = \lfloor \frac{\delta}{2} \rfloor$. Let (G, d, P) be such that (G, d) is antipodally symmetric and connected and suppose that (G, d, P) has a completion into $\mathcal{B}^{\delta}_{\infty,0,2\delta+2,2\delta+1,\mathcal{S}}$. Let $(G, \bar{d}, P)$ be its bipartite completion with a magic parameter M. Then $(G, \bar{d}, P) \in \mathcal{B}^{\delta}_{\infty,0,2\delta+2,2\delta+1,\mathcal{S}}$.*

The completion is optimal in the following sense: Let $\mathbf{G}' = (G, d', P) \in \mathcal{B}^{\delta}_{\infty,0,2\delta+2,2\delta+1,\mathcal{S}}$ be an arbitrary completion of (G, d, P) in $\mathcal{B}^{\delta}_{\infty,0,2\delta+2,2\delta+1,\mathcal{S}}$, then for every pair of vertices $u, v \in G$, $d'(u, v)$ has the same parity as $\bar{d}(u, v)$. Furthermore the completion is optimal in the following sense: For all $u, v \in P$ such that $\bar{d}(u, v) \neq M, M + 1$ we have

$$d'(u, v) \geq \bar{d}(u, v) \geq M + 1 \text{ or } d'(u, v) \leq \bar{d}(u, v) \leq M.$$

Furthermore every automorphism of (G, d, P) is also an automorphism of $(G, \bar{d}, P)$.

Proof. As in Theorem 7.4 this follows directly from the properties of the bipartite completion algorithm with parameter M for the non-antipodal class $\mathcal{A}^{\delta-1}_{\infty,0,2\delta+2,2\delta+1}$. □

As a direct consequence we obtain the following for odd diameters δ:

Corollary 7.7. *Let $3 \leq \delta < \infty$ be odd and $M = \frac{\delta-1}{2}$. Let (G, d, P) be antipodally symmetric and connected. Suppose that $\mathbf{G} = (G, d)$ has a completion into $\mathcal{A}^{\delta-1}_{\infty,0,2\delta+2,2\delta+1}$. Then there is a unique completion $(G, \bar{d}) \in \mathcal{A}^{\delta-1}_{\infty,0,2\delta+2,2\delta+1}$ that is optimal in the following sense: Let $\mathbf{G}' = (G, d') \in \mathcal{A}^{\delta-1}_{\infty,0,2\delta+2,2\delta+1}$ be an arbitrary completion of $\mathbf{G}$, then, for all $u, v \in G$ one of the following holds:*

1. *$d'(u, v) \geq \bar{d}(u, v) \geq M + 1$ or*
2. *$d'(u, v) \leq \bar{d}(u, v) \leq M$ or*
3. *$M \leq \bar{d}(u, v) \leq M + 1$*

Furthermore $\mathrm{Aut}(G, d) = \mathrm{Aut}(G, \bar{d})$.

Proof. Since δ is odd, we have that $M + (M + 1) = \delta$. Hence the optimality property for pairs $u, v \in P$, described in Theorem 7.6 transfers to all pairs $u, v \in G$ (by considering antipodal quadruples).

As in the proof of Corollary 7.5, this optimality properties imply that the completion does not depend of the choice of P. Furthermore analogue to Corollary 7.5 one can show that automorphisms of (G, d) are also automorphisms of $(G, \bar{d})$. □

We remark that in the last remaining case — bipartite antipodal spaces of even diameter — again the standard algorithm for bipartite spaces fails, as shown in the following example:

Example. Consider $\mathcal{A}^{4}_{\infty,0,10,9}$ which is an example of a class of bipartite antipodal metric spaces. Completing the graph depicted in Figure 12 using Definition 6.1 and $M = 2$ will result in non-antipodal metric space where every non-edge will be completed by 3.

7.3. Antipodality-preserving EPPA

In the following we add extra layer to the construction of Herwig and Lascar which makes sure that vertex closures are preserved. This idea is the same as in [38, 39].

Theorem 7.8. *Let $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ be an antipodal sequence of admissible parameters.*

1. *If $K_1 < \infty$ and δ is even then the class $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1} \cap \mathcal{A}_{\mathcal{S}}$ has coherent EPPA.*

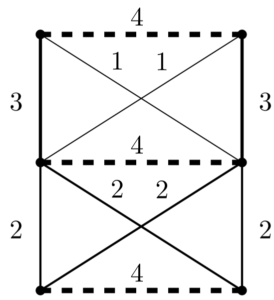

Figure 12: An example of graph which fails to be completed to $\mathcal{A}^4_{\infty,0,10,9}$ using the algorithm given by Definition 6.1.

2. *If $K_1 = \infty$ and δ is odd then the class $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1} \cap \mathcal{A}_\mathcal{S}$ has coherent EPPA.*

3. *If $K_1 < \infty$ and δ is odd then $S = \emptyset$ and the class $\mathcal{B}^\delta_{K_1,K_2,C_0,C_1}$ expanding $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ by an additional unary predicate P determining the podality (so every edge of length δ has precisely one vertex $v \in P$) has coherent EPPA.*

4. *If $K_1 = \infty$ and δ is even then $S = \emptyset$ and the class $\mathcal{B}^\delta_{K_1,K_2,C_0,C_1}$ expanding $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$ by an additional unary predicate P determining the podality has coherent EPPA.*

Proof. We first give a proof for the even diameter non-bipartite case with $\mathcal{S} = \emptyset$ and then discuss how to extend the construction for remaining cases.

1. Given $\mathbf{A} = (A, d) \in A^\delta_{K_1,K_2,C_0,C_1}$ without loss of generality we can assume that for every vertex $u \in A$ there exists unique vertex $v \in A$, with $d(u, v) = \delta$. Denote by $\mathcal{O}$ the set of obstacles of $\mathcal{A}^{\delta-1}_{K_1,K_2,C_0,C_1}$ given by Lemma 4.18. Apply Theorem 2.2 to obtain $\mathbf{B} \in \mathrm{Forb}(\mathcal{O})$ which is a coherent extension of $\mathbf{A}$ and has no homomorphic image of any structure in $\mathcal{O}$. Note that $\mathbf{B} = (B, d)$ contains edges of type δ and thus it can not be completed to a metric space in $\mathcal{A}^{\delta-1}_{K_1,K_2,C_0,C_1}$.

Now we describe edge-labelled graph $\mathbf{C} = (C, d')$. C is the set of all ordered pairs (u, v) where $u \neq v \in B$, $d(u, v) = \delta$. We set

$$d'((u,v),(u',v')) = \begin{cases} 0 & \text{if } (u,v) = (u',v') \\ \delta & \text{if } u = v', v = u' \\ d(u,u') & \text{if } (u,v,u',v') \text{ is an antipodal quadruple.} \end{cases}$$

Clearly edges of distance δ form a (complete) matching in $\mathbf{C}$.

Given $u \in A$ put $\varphi(u) = (u, v)$ where v is the unique vertex of $\mathbf{A}$ such that $d(u, v) = \delta$. It is easy to check that φ is an embedding $\mathbf{A} \to \mathbf{C}$. We show that $\mathbf{C}$ is an coherent extension of $\varphi(\mathbf{A})$.

Denote by $\pi : C \to B$ the *projection* assigning $(u, v) \mapsto u$. Let ψ be a partial isometry of $\varphi(\mathbf{A})$. Then $\psi \circ \pi$ is a partial isometry of $\mathbf{A}$. Now extend this isometry to $\bar{\psi} : \mathbf{B} \to \mathbf{B}$. It follows that the mapping $\bar{\psi}' : C \to C$ defined by $\bar{\psi}'(u, v) = (\bar{\psi}(u), \bar{\psi}(v))$ is an automorphism of $\mathbf{C}$ extending ψ.

It remains to show that there is a completion of $\mathbf{C}$ to a metric space $\mathbf{C}' \in \mathcal{A}^\delta_{K_1,K_2,C_0,C_1}$. For every pair of vertices (u, v), such that $d'(u, v) = \delta$ choose arbitrarily one of the vertices and denote by $\mathbf{C}_{\delta-1}$ the subgraph induced by C on all those vertices. As the edges of length δ form a complete matching, we know that $\mathbf{C}_{\delta-1}$ is precisely half of the graph $\mathbf{C}$ and whenever a distance is defined in $C \setminus C_{\delta-1}$ it is part of a unique antipodal quadruple.

Note that $\pi(\mathbf{C}_{\delta-1})$ is an isomorphic copy of $\mathbf{C}_{\delta-1}$ to $\mathbf{B}$ and $\mathbf{C}_{\delta-1}$ does not contain any edges of length δ. Because $\mathcal{K}$ is closed for antipodal companions, we then know that $\mathbf{C}_{\delta-1} \in \mathrm{Forb}(\mathcal{O})$. Denote by $\mathbf{C}'_{\delta-1} \in \mathcal{K}$ its completion with the magic parameter $M = \frac{\delta}{2}$ and by $\mathbf{C}' \in \mathcal{K}^\ominus$ the completion of $\mathbf{C}$ which extends $\mathbf{C}'_{\delta-1}$ antipodaly to the remaining vertices.

We proved that there is a completion of $\mathbf{C}$ into $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$, so now we can look at the completion with magic parameter M of $\mathbf{C}$ given by Theorem **??**. This completion (which in fact is equivalent to one described above) preserves all automorphisms of $\mathbf{C}$, which is what we wanted.

Now consider the case when $\mathcal{S} \neq \emptyset$. Again we use the fact that the completion algorithm will never introduce new Henson substructures and that the set of obstacles can be extended by $\mathcal{S}$.

2. Now we adjust the construction for the bipartite case of odd diameter. Consider again $\mathbf{A} = (A, d) \in A^{\delta}_{K_1,K_2,C_0,C_1}$ such that for every vertex $u \in A$ there exists unique vertex $v \in A$, $d(u, v) = \delta$. Denote by O the set of obstacles given by Lemma 6.11 for $\mathcal{A}^{\delta-1}_{K_1,K_2,C_0,C_1}$ (which is a non-antipodal bipartite amalgamation class). By the same construction as we used to build the extension $\mathbf{C}$ in the proof of Theorem 6.13 we obtain a coherent extension $\mathbf{B} \in \mathrm{Forb}(O)$ of $\mathbf{A}$ without odd cycles. Now we use the same construction as above to obtain a coherent extension $\mathbf{C} \in \mathrm{Forb}(O)$ where edges of length δ form a matching. Note that, since $\mathbf{B}$ can be chosen to be connected, we can also assume that $\mathbf{C}$ is connected. Finally we apply Theorem **??** to complete it to bipartite metric space in $\mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1}$.

3. In the third case of odd diameter non-bipartite spaces we proceed similarly to the first case, however this time with the explicit predicate P. Consider a structure $\mathbf{A} = (A, d, P) \in \mathcal{B}^{\delta}_{K_1,K_2,C_0,C_1}$, such that for every vertex $u \in A$ there exists a unique vertex $v \in A$ with $d(u, v) = \delta$. Denote by O the set of obstacles given by Lemma 4.18 for $\mathcal{A}^{\delta-1}_{K_1,K_2,C_0,C_1}$. We apply Theorem 2.2 to obtain a structure $\mathbf{B} \in \mathrm{Forb}(O)$ which is a coherent extension of $\mathbf{A}$. Without loss of generality we can assume that the predicate P in $\mathbf{B}$ picks precisely one vertex of every edge of length δ (all other edges of length δ can be removed from $\mathbf{B}$ because they are never contained in a copy of $\mathbf{A}$). Then we construct $\mathbf{C}$ in the same way as before, with π being the projection of $\mathbf{C}$ onto the P-part of $\mathbf{B}$.

Mat: Shouldn't we say what P looks like in $\mathbf{A}$?
Pat: I think it is fine — P of $\mathbf{A}$ extends to P of $\mathbf{B}$

In the last step we complete $\mathbf{C}$ according to the completion algorithm described in Definition 7.6. By Theorem 7.4 this completion preserves automorphisms and gives the desired coherent extension in $\mathcal{B}^{\delta}_{K_1,K_2,C_0,C_1}$.

4. The last case can be dealt with by the combination of case 2 and 3. □

7.4. Stationary independence relation

In this section we discuss the existence of a stationary independence relation on antipodal spaces. As one might suspect the answer is related to the question if there is a deterministic completion algorithm with a magic parameter. Our results show that if $K_1 < \infty$ and δ is odd, or if $K_1 = \infty$ and δ is even, there is no local SIR.

Theorem 7.9. *Let $(\delta, K_1, K_2, C_0, C_1, \mathcal{S})$ be a sequence of admissible antipodal parameters. Then the following holds in $\Gamma^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$:*

1. *If $K_1 < \infty$ and δ is even, there is a stationary independence relation.*
2. *If $K_1 = \infty$ and δ is odd, there is no stationary independence relation, but a local one.*
3. *If $K_1 < \infty$ and δ is odd, there is no local stationary independence relation.*
4. *If $K_1 = \infty$ and δ is even, there is no local stationary independence relation.*

Proof.

1. For $K_1 < \infty$ and even diameter δ, Theorem **??** gives us a completion algorithm with magical parameter $M = \frac{\delta}{2}$. For structures $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{A}^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ we define $\mathbf{A} \oplus_{\mathbf{C}} \mathbf{B}$ to be the space obtained by first forming the free amalgam of $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{C}$ and then forming the completion with magical parameter $M = \frac{\delta}{2}$. This operator is a canonical amalgamation operator, hence there is a stationary independence relation on $\Gamma^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ (cf. Corollary 4.17).

2. For $K_1 = \infty$ and odd diameter δ, use Theorem **??** for connected edge-labelled graphs. Then, as in the proof of Corollary 6.10 one can show, that there is a local stationary independence relation. In order to see that there is no stationary independence relation, we show that there is no canonical amalgamation in $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$ over the empty structure. For that let $\mathbf{A}$ consist of an antipodal pair a_1, a_2 and let $\mathbf{B}$ be consist of a single vertex b. If there was a canonical amalgamation operator then in $\mathbf{A} \oplus \mathbf{B}$ we have that $d(b, a_1) = \delta - d(b, a_2)$, and since δ is odd $d(b, a_1) \neq d(b, a_2)$. Hence $\{a_1\} \oplus \mathbf{B}$ and $\{a_2\} \oplus \mathbf{B}$ are non-isomorphic, which contradicts the assumption that $\oplus$ is a canonical amalgamation operator.

3. Next assume that $K_1 < \infty$, δ is odd, and let $M = \lfloor \frac{\delta}{2} \rfloor$. We are going to show that there is no local canonical amalgamation operation on $\mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$. Suppose for a contradiction that there is such an operator $\oplus$. Let $\mathbf{C} = (\{c_0, c_1\}, d)$ with $d(c_0, c_1) = M$, let $\mathbf{A} = (\{c_0, c_1, a_0, a_1\}, d)$ with $d(a_0, a_1) = \delta$, $d(a_i, c_i) = M$ and $d(a_i, c_{1-i}) = M + 1$, where $i \in \{0, 1\}$. Note that the triangles a_0, c_0, c_1 and a_1, c_1, c_0 are isomorphic. Also let $\mathbf{B} = (\{c_0, c_1, b\}, d)$ with $d(b, c_0) = d(b, c_1) = M$. Let us consider the canonical amalgam $\mathbf{A} \oplus_\mathbf{C} \mathbf{B}$. Note that in this amalgam b cannot be identified with a_0 or a_1, so we can write $\mathbf{A} \oplus_\mathbf{C} \mathbf{B}$ as $(\{a_0, a_1, b, c_0, c_1\}, d)$ for some distance function d. By the antipodality we have that $d(a_0, b) = \delta - d(a_1, b)$, hence $d(a_0, b) \neq d(a_1, b)$. By the monotonicity of $\oplus$, we have $(\{a_0, b, c_0, c_1\}, d) = \{a_0, c_0, c_1\} \oplus_\mathbf{C} \mathbf{B}$ and $(\{a_1, b, c_1, c_0\}, d) = \{a_1, c_1, c_0\} \oplus_\mathbf{C} \mathbf{B}$. But $(\{a_0, b, c_0, c_1\}, d)$ and $(\{a_1, b, c_0, c_1\}, d)$ are not isomorphic, which contradicts the assumption that $\oplus$ is an amalgamation operation.

4. Finally let us assume that $K_1 = \infty$ and δ is even. Then let us consider structures $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{A}^\delta_{K_1,K_2,C_0,C_1,\mathcal{S}}$, such that $\mathbf{C} = (\{c\}, d)$, $\mathbf{B} = (\{b_1, b_2, c\}, d)$ with $d(b_1, b_2) = \delta$ and $d(b_1, c) = d(b_2, c) = \frac{\delta}{2}$ and $\mathbf{A} = (\{a, c\}, d)$ with $d(a, c) = 1$. Assume that there is a local canonical amalgamation operation $\oplus$, then let us consider the canonical amalgam $\mathbf{A} \oplus_\mathbf{C} \mathbf{B}$. By the monotonicity of $\oplus$ we must have $d(a, b_1) = d(a, b_2)$ in this amalgam. Since (b_1, b_2) is an antipodal pair this implies that $d(a, b_1) = d(a, b_2) = \frac{\delta}{2}$. However then $\mathbf{A} \oplus_\mathbf{C} \mathbf{B}$ cannot be a bipartite distance graph, since then (a, c, b_1) is a triangle of odd perimeter $\delta + 1$, a contradiction. □

8. Classes with infinite diameter

Classes of infinite diameter have two basic types—the 3-constrained cases and tree-like graphs.

8.1. 3-constrained spaces of infinite diameter

The admissible numerical parameters with $\delta = \infty$ are

1. (I) $K_1, C_0, C_1 = \infty$, $K_2 = 0$ (the generic bipartite metric space)
2. (II) $1 \leq K_1 < \infty$, $C_0, C_1, K_2 = \infty$ (the generic metric spaces without short odd cycles)

Theorem 8.1. *The class $\overrightarrow{\mathcal{A}}^\infty_{\infty,0,\infty,\infty}$ of convex orderings of $\mathcal{A}^\infty_{\infty,0,\infty,\infty}$ with an additional unary predicate determining the bipartition is Ramsey and has the expansion property.*

Proof. For every choice of finite $\mathbf{A}, \mathbf{B}$ denote by δ the maximal distance in $\mathbf{B}$. Then apply Theorem 6.12 for $\overrightarrow{\mathcal{A}}^\delta_{\infty,0,C_0,2\delta+1}$ where $C_0 > 3\delta + 1$ even to obtain $\mathbf{C} \longrightarrow (\mathbf{B})^\mathbf{A}_2$. It is easy to see that $\mathbf{C} \in \overrightarrow{\mathcal{A}}^\infty_{\infty,0,\infty,\infty}$.

The expansion property follows again by the standard argument. □

Theorem 8.2. *For every $1 \leq K_1 \leq \infty$ the class $\overrightarrow{\mathcal{A}}^\infty_{K_1,\infty,\infty,\infty}$ of free orderings of $\mathcal{A}^\infty_{K_1,\infty,\infty,\infty}$ is Ramsey and has the expansion property.*

Proof. Analogously to the previous proof, for every finite $\mathbf{A}, \mathbf{B}$ we choose δ to be the maximal distance in $\mathbf{B}$ and reduce to Theorem 4.19. □

Theorem 8.3. *The classes $\mathcal{A}^\infty_{K_1,\infty,\infty,\infty}$, $1 \leq K_1 \leq \infty$, and $\mathcal{A}^\infty_{\infty,0,\infty,\infty}$ have coherent EPPA.*

Proof. Again, this follows by a reduction to Theorems 4.20 and 6.13. □

The stationary independence relation follows from the existence of completion algorithm which adds the shortest path distances, as discussed in Section 1.1. From this we immediately get:

Theorem 8.4. *The classes $\mathcal{A}^\infty_{K_1,\infty,\infty,\infty}$, $1 \leq K_1 \leq \infty$, and $\mathcal{A}^\infty_{\infty,0,\infty,\infty}$ have no stationary independence relations but local stationary independence relations that can be defined by means of the shortest path completion algorithm.*

8.2. Tree-like graphs

Recall Definition 1.1 of tree-like graphs and denote by $\mathcal{A}_{T_{m,n}}$ the age of metric space associated with the tree-like graph $T_{m,n}$.

Theorem 8.5. *For every $2 \leq m, n \leq \infty$ the class $\mathcal{A}_{T_{m,n}}$ has no EPPA.*

Proof. For every $\mathbf{F} \in \mathcal{A}_{T_{m,n}}$, let us call a vertex of $\mathbf{F}$ a *non-leaf* if it is contained in at least two blocks (maximal cliques) of $\mathbf{F}$ and *leaf* otherwise. Note that every $\mathbf{F} \in \mathcal{A}_{T_{m,n}}$ contains a leaf. Now suppose for a contradiction that $\mathcal{A}_{T_{m,n}}$ has EPPA, and consider any structure $\mathbf{A} \in \mathcal{A}_{T_{m,n}}$ which contains a non-leaf. Given a finite extension $\mathbf{B} \in \mathcal{A}_{T_{m,n}}$ of $\mathbf{A}$, we can assume without loss of generality that for every vertex v of $\mathbf{B}$ there exists $\phi \in \mathrm{Aut}(\mathbf{B})$ such that $\phi(v) \in A$. By the extension property it then follows that the graph $\mathbf{B}$ is vertex transitive and thus either every vertex of $\mathbf{B}$ is a leaf or a non-leaf, which contradicts to the assumption that $\mathbf{A}$ contains a non-leaf. $\square$

Theorem 8.6. *For every $2 \leq m, n \leq \infty$ the class $\mathcal{A}_{T_{m,n}}$ has no precompact Ramsey expansion.*

Proof. Assume for a contradiction that there is a precompact Ramsey expansion $\mathcal{K}$ of $\mathcal{A}_{T_{m,n}}$. Denote by $\mathbf{A}_1^+, \mathbf{A}_2^+, \dots \mathbf{A}_n^+$ all structures in $\mathcal{K}$ with one vertex. Let $\mathbf{B}$ be a path of edges in distance 2 with $n+1$ vertices completed to a metric space in $\mathrm{Age}(T_{m,n})$. Denote by $\mathbf{B}^+$ its expansion in $\mathcal{K}$. There is some $\mathbf{A}_i$, $1 \leq i \leq n$, with at least two copies in $\mathbf{B}^+$. Denote by $\mathbf{B}_2^+$ the substructure induced by $\mathbf{B}^+$ on those two copies and by ℓ their distance.

By the Ramsey property, there is $\mathbf{C}^+ \in \mathcal{K}$ such that $\mathbf{C}^+ \longrightarrow (\mathbf{B}_2^+)^{\mathbf{A}_i^+}_{\ell+1}$ in $\mathcal{K}$. Without loss of generality we can assume that $\mathbf{C}^+$ is an expansion of a connected fragment of $T_{m,n}$. Fix a vertex $v \in \mathbf{C}^+$ and colour the vertices $u \in \mathbf{C}^+$ with $d(u, v) \mod \ell + 1$. Denote by u_1, u_2 the vertices of the monochromatic copy of $\mathbf{A}_i^+$ in $\mathbf{C}^+$, ensuring that u_1 is closer to v than u_2. Notice that between any pair of vertices at distance greater than 1 there is precisely one path, so it follows that $d(v, u_2) = d(v, u_1) + \ell$. By the choice of colouring this pair is not monochromatic, contradicting the choice of $\mathbf{C}^+$. $\square$

Lemma 8.7. *For all $2 \leq m, n \leq \infty$ with $m < \infty$ or $n \neq 2, 3, \infty$ there is no local stationary independence relation on the metric space associated with the tree-like graph $T_{m,n}$*

Proof. For a contradiction, assume that there is a local stationary independence relation $\mathop{\smile\kern-0.65em\raise0.5ex\hbox{$|$}}$. By Theorem 4.16 this is equivalent to the existence of a local canonical amalgamation $\oplus$ on the $\mathcal{A}_{T_{m,n}}$.

First let us look at the case where $m < \infty$. Then let $\mathbf{C}$ consist of a single vertex c and let $\mathbf{A} = \{c, a_1, a_2, \dots, a_m\}$ such that $d(c, a_i) = 1$ and $d(a_i, a_j) = 2$ for all $i \neq j$. And let $\mathbf{B} = \{b, c\}$ with $d(b, c) = 1$. Then, by the definition of $(T_{m,n}, d)$, in the amalgam $\mathbf{A} \oplus_{\mathbf{C}} \mathbf{B}$ there has to be an index i, such that $d(b, a_i)$ is 1 or 0 and $d(b, a_j) = 2$ for all $j \neq i$. Thus $\{a_i, c\} \oplus_{\mathbf{C}} \mathbf{B}$ and $\{a_j, c\} \oplus_{\mathbf{C}} \mathbf{B}$ are non-isomorphic for $j \neq i$. But this contradicts the assumption that $\oplus$ is an amalgamation operator.

Next assume that $n \notin \{2, 3, \infty\}$. Let $\mathbf{C} = \{c_1, c_2\}$ be such that $d(c_1, c_2) = 1$, let $\mathbf{B} = \{c_1, c_2, b\}$ be a clique of size 3 and let $\mathbf{A} = \{c_1, c_2, a_1, \dots, a_{n-2}\}$ be a clique of size n. Then, by definition of $\mathcal{A}_{T_{m,n}}$, in the amalgam $\mathbf{A} \oplus_{\mathbf{C}} \mathbf{B}$ the vertex b has to identified with one of the vertices a_i in $\mathbf{A}$. By the monotonicity of $\oplus$ we have $\{a_i, c_1, c_2\} \oplus_{\mathbf{C}} \mathbf{B}$ is a clique of size 3, but for every $j \neq i$, $\{a_j, c_1, c_2\} \oplus_{\mathbf{C}} \mathbf{B}$ is a 4-clique. This contradicts the assumption that $\oplus$ is an amalgamation operator. $\square$

Surprisingly, for the remaining cases there exists no SIR, but there is a local SIR. In order to show this, we first need a better understanding how the metric space $(T_{n,m}, d)$ relates to the underlying tree-like graph.

Definition 8.1. Let (G, E) be a graph, inducing the graph metric (G, d), and let $\mathbf{A} = (A, d)$ be a subspace of (G, d). Let $a, b \in A$, then we say that a path $a = y_0, y_1, \dots, y_k = b$ in (G, E) *witnesses the distance* $d(a, b)$, if $d(a, b) = k$.

Lemma 8.8. *Let $\mathbf{A}$ be a subspace of $(T_{n,m}, d)$ and let (G_A, E) be a subgraph of $T_{n,m}$, consisting of paths witnessing the distances in $\mathbf{A}$. Then, (G_A, E) is uniquely determined by $\mathbf{A}$. Also, if $\mathbf{A} \cong \mathbf{B}$, then $(G_A, E) \cong (G_B, E)$.*

Proof. It is enough to show that for every pair u, v in $\mathbf{A}$, there is exactly one path in $T_{n,m}$ witnessing its distance $d(u, v) = c$. Then clearly (G_A, E) is uniquely determined by $\mathbf{A}$; the second statement follows from the homogeneity of $(T_{n,m}, d)$.

For a contradiction, assume that $c \geq 2$ and that there are two paths $u = x_0, x_1, x_2, \dots, x_c = v$ and $u = y_0, y_1, y_2, \dots, y_c = v$. Since the two paths are not equal, there have to be indices $i < k < c$ such that $x_i = y_i$, $x_{i+1} \neq y_{i+1}, \dots, x_k \neq y_k$ and $x_{k+1} = y_{k+1}$. Then, by definition of $T_{n,m}$, the set $\{x_i, x_{i+1}, \dots x_k\} \cup \{y_{i+1}, \dots y_k\}$ has to be a clique. But this implies that $d(u, v) < c$, which contradicts our assumptions. □

Theorem 8.9. *For every m, n, the metric space associated with tree-like graph $T_{m,n}$ has no stationary independence relation. It has a local stationary independence relation if and only if $m = \infty$ and $n \in \{2, 3, \infty\}$.*

Proof. By Lemma 8.7 the statement about local SIR holds for all cases where $m \neq \infty$ or $n \notin \{2, 3, \infty\}$.

In order to prove that $(T_{m,n}, d)$ has no SIR we show that there is no canonical amalgamation operator on $\mathcal{A}_{T_{m,n}}$ (cf. Theorem 4.16). Assume there is such an operator $\oplus$. Let $\mathbf{A} = \{a_1, a_2, a_3\}$ with $d(a_1, a_2) = d(a_2, a_3) = 1$ and $d(a_1, a_3) = 2$ and let $\mathbf{B}$ consist of a single point b. We claim that then, no matter how the amalgam $\mathbf{A} \oplus \mathbf{B}$ is formed, we have $d(b, a_1) \neq d(b, a_2)$ or $d(b, a_1) \neq d(b, a_3)$. In order to prove this claim, let us assume that $d(b, a_1) = d(b, a_3)$. By Lemma 8.8 the graph witnessing the distances in $\mathbf{A} \oplus \mathbf{B}$ has to consist of a path from b to a_2 and the two edges $d(a_2, a_3) = d(a_2, a_1) = 1$. Hence $d(b, a_2) = d(b, a_1) - 1$, which proves our claim. By the claim and the monotonicity of $\oplus$, $\{a_1\} \oplus \mathbf{B}$ is non-isomorphic to $\{a_i\} \oplus \mathbf{B}$ for some $i \in \{2, 3\}$. But this contradicts to the assumption that $\oplus$ is an amalgamation operator.

It is only left to show that for $m = \infty$ and $n \in \{2, 3, \infty\}$, there is a local stationary independence relation on $(T_{m,n}, d)$. We do so again by finding a local canonical amalgamation operator. So let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be non-empty substructures of $(T_{m,n}, d)$ such that $e_1 : \mathbf{C} \to \mathbf{A}$ and $e_2 : \mathbf{C} \to \mathbf{B}$ are embeddings. By Lemma 8.8 we can assume that every distance in the structures $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is witnessed by a path (so informally we can think about $\mathbf{A}, \mathbf{B}, \mathbf{C}$ as the underlying tree-like graphs instead of the metric spaces).

If $m = \infty$ and $n = 2$ we define $\mathbf{A} \oplus_{\mathbf{C}} \mathbf{B}$ to be the free amalgam of the trees $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{C}$. This graph is again a tree and thus in $T_{2,\infty}$. It is easy to see that this amalgamation operator is associative and monotone.

If $m = \infty$ and $n = 3$ we construct $\mathbf{A} \oplus_{\mathbf{C}} \mathbf{B}$ again by forming first the free amalgam of the graphs $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{C}$. This free amalgam might contain c_1, c_2, a, b, such that a, c_1, c_2 is a 3-clique and b, c_1, c_2 is a 3-clique, but there is no edge between a and b. In this case we identify a and b. Again this amalgamation operator is associative and monotone.

If $m = \infty$ and $n = \infty$ we construct $\mathbf{A} \oplus_{\mathbf{C}} \mathbf{B}$ by forming first the free amalgam of the graphs $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{C}$. Then we add edges to complete all the subgraphs $A' \cup B'$ with $A' \subseteq A$ and $B' \subseteq B$, such that A' and B' are cliques of size ≥ 3 and share at least one edge. Again this amalgamation operator is associative and monotone. □

9. Corollaries

While the main focus on our work on the combinatorial properties of metrically homogeneous graphs, let us briefly discuss the corollaries of our results on the automorphism group of the Fraïssé limits. Many of those results can now be considered folklore and this section is not meant to be exhaustive. We refer reader to a recent survey [21] and thesis [16] which discuss many of the topics in a greater detail.

9.1. Ample generics

The automorphism group of the Fraïssé limits of amalgamation classes in the catalogue can be seen as Polish groups when endowed with the pointwise convergence topology. A Polish group has *generic automorphisms* if it contains a comeagre conjugacy class. A Polish group has *ample generics* if it has a comeagre diagonal conjugacy class in every dimension.

Known examples of structures with ample generics include ω-stable ω-categorical structures [14], the random graph [14, 7], the homogeneous K_n-free graph [40], the rational Urysohn space [9, 10], free homogeneous structures over finite relational languages [15] and Philip Hall's locally finite universal group [16].

Our study of coherent EPPA is directly motivated by the following result:

Theorem 9.1 (Siniora-Solecki [16]). *Suppose that* $\mathbf{M}$ *is a homogeneous structure such that* $\mathrm{Age}(\mathbf{M})$ *has both EPPA and APA . Then* $\mathbf{M}$ *has ample generics.*

I don't think APA is defined in this paper. What is it? -Mike

Where we say a class $\mathcal{K}$ of finite structures has the *amalgamation property with automorphisms (APA)* if whenever $\mathbf{A}, \mathbf{B}_1, \mathbf{B_2} \in \mathcal{K}$ with embeddings $\alpha_1 : \mathbf{A} \to \mathbf{B}_1$ and $\alpha_2 : \mathbf{A} \to \mathbf{B}_2$ then there is a structure $\mathbf{C} \in \mathcal{K}$ with embeddings $\beta_1 : \mathbf{B}_1 \to \mathbf{C}$ and $\beta_2 : \mathbf{B}_2 \to \mathbf{C}$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ and whenever $f \in \mathrm{Aut}(\mathbf{B}_1)$ and $g \in \mathrm{Aut}(\mathbf{B}_2)$ such such that $f \circ \alpha_1(\mathbf{A}_1) = \alpha_1(\mathbf{A})$, $g \circ \alpha(\mathbf{A})$, and for every $a \in \mathbf{A}$ we have $\alpha_1^{-1} \circ f \circ \alpha_1(a) = \alpha_2^{-1} \circ g \circ \alpha_2(a)$, then there exists $h \in \mathrm{Aut}(\mathbf{C})$ which extends $\beta_1 \circ f \circ \beta_1^{-1} \cup \beta_2 \circ g \circ \beta_2^{-1}$. Observe that APA follows from our canonical amalgamation and the automorphism preservation lemma for all classes where we shown the coherent EPPA. As a corollary of Theorem 1.2 we thus extended the list of known structures with ample generics by many new examples.

Ample generics are a powerful tool to prove additional properties. Among multiple consequences of the ample generic we list the following:

Theorem 9.2 (Kechris-Rosendal [6]). *Suppose that G is a Polish group with ample generics. Then G has the small index property*

Theorem 9.3 (Kechris-Rosendal [6]). *Suppose that* $\mathbf{M}$ *is an* ω*-categorical structure with ample generics. Then* $\mathrm{Aut}(\mathbf{M})$ *has uncountable cofinality and 21-Bergman property.*

Theorem 9.4 (Siniora [16]). *Suppose that* $\mathbf{M}$ *is an* ω*-categorical structure with ample generics. Then* $\mathrm{Aut}(\mathbf{M})$ *has Serre's property (FA).*

Consequently we also gave a number of new examples of structures whose automorphism group has small index property, uncountable cofinality, 21-Bergman property and Serre's property (FA).

9.2. Amenability and unique ergodicity

Let $G = \mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$, endowed with the topology of pointwise convergence. In the following we set up the machinery needed to discuss when G is amenable (its universal minimal flow has a G-invariant measure) or uniquely ergodic (a unique such measure). See [21] and [41] for a more in depth discussion of the definitions and their history. We now state precise definitions and lemmas which capture the idea that "amenability and unique ergodicity can sometimes be shown by counting finite quantities related to the number of Ramsey expansions of some finite structures." We have in mind the example that arbitrary linearly ordered metric graphs (with a fixed set of parameters) is a precompact Ramsey expansion of the corresponding metric graphs (see Theorem 1.1).

Let $L \subseteq L^+$ be languages. Let $\mathcal{K}$ and $\mathcal{K}^+$ be classes of structures in L and L^+ respectively such that $\mathcal{K}^+|L = \mathcal{K}$ (i.e. $\mathcal{K}^+$ is an expansion of $\mathcal{K}$). If $\mathbf{B} \in \mathcal{K}^+$ then we write $\mathbf{B}^+ = (\mathbf{A} \oplus \mathbf{A}^+)$ where $\mathbf{A} = \mathbf{B}^+|L$ and $\mathbf{A}^+ = \mathbf{B}^+|(L^+ \setminus L)$. Colloquially, "$\mathbf{A}$ is the old stuff, and $\mathbf{A}^+$ is the new stuff" when we use the representation $(\mathbf{A} \oplus \mathbf{A}^+)$.

Let $\mathbf{A} \leq \mathbf{B}$ be structures in $\mathcal{K}$ and let $(\mathbf{A} \oplus \mathbf{A}^+) \in \mathcal{K}^+$. Then we write:

$$\mathrm{Nexp}_{\mathcal{K}^+}(\mathbf{A}^+, \mathbf{B}) := \left|\{\mathbf{B}^+ : (\mathbf{A} \oplus \mathbf{A}^+) \leq (\mathbf{B} \oplus \mathbf{B}^+) \in \mathcal{K}^+\}\right|,$$

which is the number of expansions of $\mathbf{B}$ in $\mathcal{K}^+$ that extend $(\mathbf{A} \oplus \mathbf{A}^+)$. If there is no confusion then we write $\mathrm{Nexp}(\mathbf{A}^+, \mathbf{B})$.

The following black-box lemma reduces checking amenability to a counting argument.

Lemma 9.5 (Angel-Kechris-Lyons [41], Pawliuk-Sokić [42]). *Let* $\mathcal{K}^+$ *be a precompact Ramsey expansion of* $\mathcal{K}$*. If for every* $\mathbf{A} \leq \mathbf{B}$ *in* $\mathcal{K}$ *and every* $(\mathbf{A} \oplus \mathbf{A}')$, $(\mathbf{A} \oplus \mathbf{A}'') \in \mathcal{K}^+$ *we have*

$$\mathrm{Nexp}(\mathbf{A}', \mathbf{B}) = \mathrm{Nexp}(\mathbf{A}'', \mathbf{B}) \tag{1}$$

then $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ *is amenable.*

In particular, this will happen if the values in equation (1) only depend on the isomorphism type of $\mathbf{A}$ *and* $\mathbf{B}$*.*

For fixed structures $\mathbf{A}$ and $\mathbf{B}$ with expansions $(\mathbf{A} \oplus \mathbf{A}^+)$ and $(\mathbf{B} \oplus \mathbf{B}^+)$, define

$$\mathrm{Nemb}(\mathbf{A}, \mathbf{B}) := |\{\mathbf{A} : \mathbf{A} \leq \mathbf{B}\}|\,.$$

Let $\mathbf{H} \in \mathcal{K}$ with $|\mathbf{H}| = k$ and $\mathbf{G}$ is a random element of $\mathcal{K}$ with $|\mathbf{G}| = n$ (typically $k << n$). Let

$$f(\mathbf{G}) := \frac{\mathrm{Nemb}(\mathbf{H}, \mathbf{G})}{\mathbb{E}[\mathrm{Nemb}(\mathbf{H}, \mathbf{G})]}, \qquad f^+(\mathbf{G}) := \frac{\mathrm{Nexp}(\mathbf{H}^+, \mathbf{G})}{\mathbb{E}[\mathrm{Nemb}(\mathbf{H}, \mathbf{G})]},$$

where $\mathbb{E}$ is the expected value. See section 7.2 of [42] for a more in depth discussion of these notions.

The following black-box lemma reduces checking unique ergodicity to counting three quantities.

Lemma 9.6 (Angel-Kechris-Lyons [41], Pawliuk-Sokić [42])**.** *Using the notation defined above, suppose that* $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ *is amenable, that changing* $\mathbf{G}$ *by a single edge changes f and f^+ by no more than $O(\frac{1}{n^2})$, and that the number of expansions of* $\mathbf{G} \leq O((n!)^k)$*. Then* $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ *is uniquely ergodic.*

In general these two lemmas are proved by explicit computations. While the computations asked for by Lemma 9.6 are straightforward, it is often tedious to set up and the quantities are somewhat opaque. So instead of writing down these computations for metric graphs, we will leverage computations that have already been made for unlabelled graphs. The next lemma makes this precise.

Lemma 9.7. *Let $\mathcal{K}$ and $\mathcal{F}$ be Fraïssé classes in signatures $L \subseteq F$ respectively. Let π be the map that forgets the extra structure in $F \setminus L$. Suppose that $\mathcal{K}^+$ and $\mathcal{F}^+$ are precompact Ramsey expansions of $\mathcal{K}$ and $\mathcal{F}$ respectively, such that $\pi[\mathcal{F}^+] = \mathcal{K}^+$.*

1. *If amenability of* $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ *was shown by counting (i.e. Lemma 9.5), then* $\mathrm{Aut}(\mathrm{Flim}(\mathcal{F}))$ *is amenable by counting.*
2. *If unique ergodicity of* $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ *was shown by counting (i.e. Lemma 9.6), then* $\mathrm{Aut}(\mathrm{Flim}(\mathcal{F}))$ *is uniquely ergodic by counting.*

Denote by $G = \mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$, endowed with the pointwise convergence topology.

Question 9.8. *If $\mathcal{K}$ is a tree-like class of metric graphs, is G amenable?*

Denote by $G = \mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$, endowed with the pointwise convergence topology.

Theorem 9.9. *If $\mathcal{K}$ is a primitive 3-constrained class of metric graphs, then G is amenable and uniquely ergodic.*

Proof. Since we have shown that $\overrightarrow{\mathcal{K}}$ is a precompact Ramsey expansion of $\mathcal{K}$ (see Theorem 4.19), a direct, counting proof is given by Angel, Kechris and Lyons [41]. Furthermore, this expansion guarantees that G is uniquely ergodic.

Alternatively, amenability of G will follow from the fact that $\mathcal{K}$ has the EPPA, although this is not enough for unique ergodicity. See Kechris and Rosendal [6]. □

Denote by $G = \mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$, endowed with the pointwise convergence topology.

Theorem 9.10. *If $\mathcal{K}$ is a bipartite 3-constrained class of metric graphs, then G is amenable and uniquely ergodic.*

Proof. Amenability of G follows from the fact that $\mathcal{K}$ has the EPPA, although this is not enough for unique ergodicity. See Kechris and Rosendal [6].

Alternatively, we have shown that adding linear orders that are convex with respect to the equivalence classes of even distances is a precompact Ramsey expansion of $\mathcal{K}$ (See Theorem 6.12). A direct counting argument for Lemma 9.5, is

$$\mathrm{Nexp}(\mathbf{A}', \mathbf{B}) = 2! \cdot \frac{|B_1|!|B_2|!}{|A_1|!|A_2|!}$$

where B_1, B_2 are the equivalence classes of even distances of $\mathbf{B}$, and A_1, A_2 are the equivalence classes of even distances of $\mathbf{A}$, and $\mathbf{B}$ has two equivalence classes. This computation appears as Theorem 4.2 in [42].

The computation for unique ergodicity and lemma 9.7 appears as Theorem 8.2 in [42]. □

Denote by $G = \mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$, endowed with the pointwise convergence topology.

I think this needs a refinement for the case which needs a predicate. Mike: Indeed!

Theorem 9.11. *If $\mathcal{K}$ is an antipodal class of metric graphs, then G is amenable and uniquely ergodic.*

Proof. Amenability of G follows from the fact that $\mathcal{K}$ has the EPPA, although this is not enough for unique ergodicity. See Kechris and Rosendal [6].

Alternatively, we have shown that adding linear orders that are convex with respect to the δ-equivalence classes is a precompact Ramsey expansion of $\mathcal{K}$ (See Theorem 1.1). A direct counting argument for Lemma 9.5, is

$$\mathrm{Nexp}(\mathbf{A}', \mathbf{B}) = \frac{b!}{a!} \cdot 2^{b-a},$$

where b is the number of δ-equivalence classes of $\mathbf{B}$, a is the number of equivalence classes of $\mathbf{A}$, and all equivalence classes have exactly 2 elements. This computation appears as Theorem 4.3 in [42].

The computation for unique ergodicity and Lemma 9.7 appears as Theorem 8.3 in [42]. □

10. Conclusion and open problems

1. The last remaining open case for EPPA of the known metrically homogeneous graphs is already stated as Problem 1.3: we do not know whether the requirement of a predicate P denoting the podality in Theorem 7.8 is necessary. In its easiest instance, the class $\mathcal{A}^3_{0,10,7,2}$ of all antipodal metric spaces of diameter 3, the metric spaces can be equivalently interpreted as double-covers of complete graphs. By *double-cover* of $\mathbf{K}_n$ we mean a graph $\mathbf{G}$ with a 2–to–1 covering map from the vertices of $\mathbf{G}$ to those of $\mathbf{K}_n$, so that each edge of $\mathbf{K}_n$ is covered by two edges of $\mathbf{G}$. There are two double-covers of $\mathbf{K}_3$, a pair of triangles and a hexagon.

It is a well known fact that double-covers of complete graphs correspond to *two-graphs* (3-regular hypergraphs where every subhypergraph with 4 vertices contains an even number of hyperedges): given a double-cover $\mathbf{G}$ of $\mathbf{K}_n$ put vertices $a, b, c \in K_n$ to a hyper-edge if and only if the corresponding double-cover is a hexagon.

Two-graphs has been extensively studied since 1960s [**? ?**] and many of those results are relevant to our problem. Because the correspondence between antipodal metric graphs and the underlying two-graphs is automorphism-preserving, by constructing a coherent extension in $\mathbf{A} \in \mathcal{A}^3_{0,10,7,2}$ we would also construct a coherent extension of its associated two-graph. Extensions can be assumed to be vertex transitive (because every vertex may be assumed to be contained in the copy) and construction of vertex-transitive (or regular) two-graphs is related to construction of strongly regular graphs and Taylor graphs. This is a well established topic of algebraic graph theory and those graphs are rare. Having a positive answer to the problem of EPPA of the odd non-bipartite antipodal graphs would thus require a construction of strongly regular graphs with even stronger symmetry assumptions and thus seems difficult.

Equivalently we can formulate Problem 1.3 as:

Problem 10.1. *Does the class of all finite two-graphs have coherent EPPA?*

This question was already asked by Siniora [16]. Our work can be seen as a further motivation to seek an answer for this problem. A positive answer to this problem would allow us to obtain coherent EPPA for the odd diameter antipodal metric spaces. A negative answer would show that our results are actually complete under an assumption of validity of Conjecture 3.1.

2. The classification of metrically homogeneous graphs is not a classification of metric spaces with integer distances, because it includes the additional requirement of containing all *geodesics*, i.e all triangles with edge lengths a, b and $|a - b|$. Our completion algorithm does not really rely on this requirement (though it is implicitly used in the definition of time function). It would be interesting to give more general characterisation of classes where such approach works, possibly including non-binary relations in sense of homogeneizations defined in [5].

3. One of our original motivations for investigating the problem was the problem stated by Lionel Nguyen Van Thé about the Ramsey expansion of the class of affinely independent Euclidian metric spaces [26]. Our techniques do not seem to generalise to this setting, however it seems more clear that the lack of (local) canonical amalgamation is one of the main obstacles. It would be interesting to identify classes of a more combinatorial nature which also expose such problems as an additional step in this direction.

4. The stationary independence relation can be used to show the simplicity of the automorphism group [28, 30]. For this the maximally moving automorphisms needs to be described in terms of products of conjugates which is a logical next step after completing Theorem 1.4.

5. In terms of of dynamics, we leave open the question as to whether the automorphism groups of the tree-like graphs are amenable or uniquely ergodic. Since the tree-like graphs do not have a Ramsey precompact expansion, the KPT correspondence cannot be used as it. So showing non amenability would likely go through exhibiting a compact space on which the automorphism group acts freely.

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Shortly before the work on writing this paper was finished we were informed that similar results on finite set of obstacles, Ramsey property and ample generics were also obtained by Rebecca Coulson and Gregory Cherlin and will appear in [**?**]. The Ramsey expansion of space $\mathcal{A}^3_{3,8,9,2}$ was also independently obtained by Miodrag Sokić [**?**].

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