Complexity of the group isomorphism problem

Jaroslav Hančl*, Matěj Konečný*, Jana Novotná*, Jakub Pekárek*, *Radim Předstíraný* †, Václav Rozhoň*, Jakub Svoboda*, Štěpán Šimsa*,
Advisor: Periklis Papakonstantinou♡

*Charles University, †2017, ♡Rutgers University

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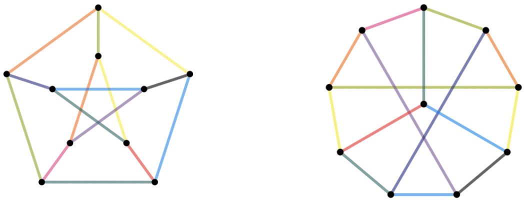
Graph isomorphism

- Given two graphs decide whether their structure is the same.



Source: `https://www.quantamagazine.org/20151214-graph-isomorphism-algorithm/`

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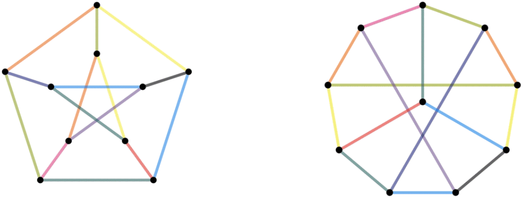

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- Graph isomorphism is in NP.
- But is believed to be easier, maybe even in P.

Theorem (Graph isomorphism is easy).

If graph isomorphism is NP-complete, something weird happens.

Graph isomorphism

Graph isomorphism is easy

- Graph isomorphism is in NP.
- Graph nonisomorphism is in IP.
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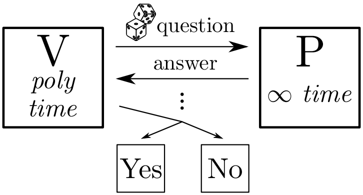
IP protocol

- Protocol between the verifier (us, polynomial time bound) and the prover (computationally unbounded).
- We send a question to the prover (with use of randomness), he sends us an answer, this repeats several times.
- The prover is proving that the two graphs or not isomorphic (even if in fact they are isomorphic).
- We want to output the correct answer with high probability.



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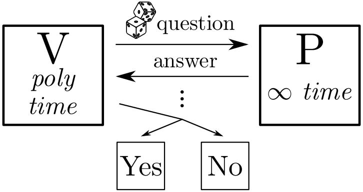

Graph nonisomorphism is in IP

- We select a random graph out of the two and a random permutation of vertices.
- We send the permuted graph to the prover.
- Prover says what graph we picked in the beginning.
- If graphs were isomorphic, with $p = \frac{1}{2}$ we will know that.
- Repeat protocol to amplify probability.

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Graph nonisomorphism is in AM

- AM is like IP but the prover knows our randomness.
- The proof is more complicated but based on the proof for the IP protocol.

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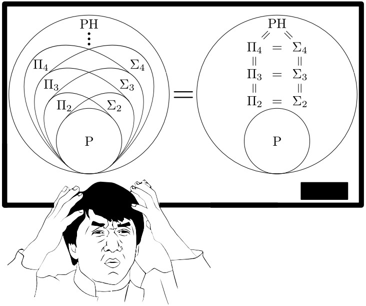

Theorem (Graph isomorphism is easy).

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Conjecture (Group isomorphism is easy).

If group isomorphism is NSC_2-complete, something weird happens.

- NSC_2 is the same as NP but the space is restricted to only $\mathcal{O}(\log^2 n)$.

Graph isomorphism vs Group isomorphism

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Group isomorphism is easy

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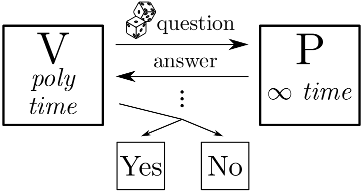

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Graph isomorphism vs Group isomorphism

- For storing a random permutation we need asymptotically $n \log(n)$ space.
- So, instead we sample randomly a set of size $100 \cdot \log(n)$ that fits in our $\mathcal{O}(\log^2(n))$ space.
- For groups, this set is (with probability at least 99%) "generating", i.e. any element of the group can be represented using only elements from this set.
- We need a deterministic process that creates a permutation of vertices based on this selected set. It must not depend on the way the groups are given to us on the input.

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Why the last step does not work

Where is $\mathrm{AM}_{\mathrm{SC}_2}$ in the polynomial hierarchy?

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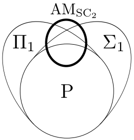

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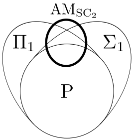

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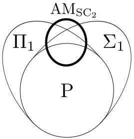

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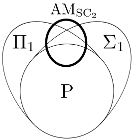

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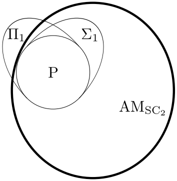

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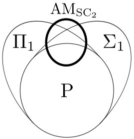

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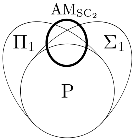

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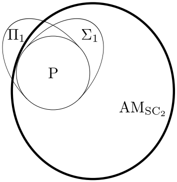

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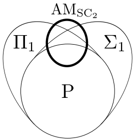

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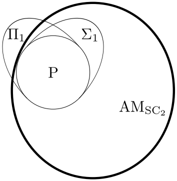

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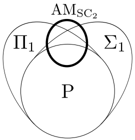

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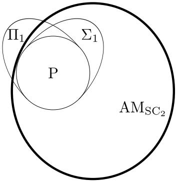

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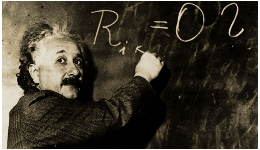

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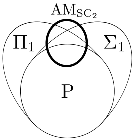

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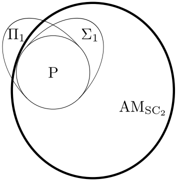

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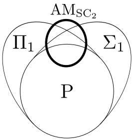

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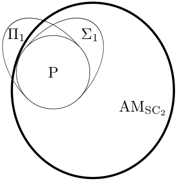

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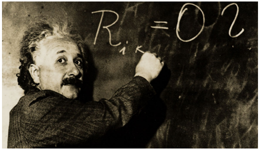

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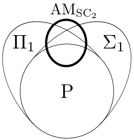

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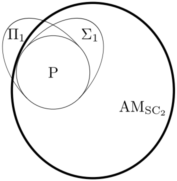

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