REU 2017 – Combinatorial Double Counting and “Not Counting”

Note. Don't spend too much time on combinatorial identities!

Note. Easy problems are 1-5, 7-10, 11, 15-18, (20)

Note. Nice problems are (in order of niceness): 23, 17, 24, 20, 22, 19

Motivation Problems

Problem 1. Every person at REU is part of three problem solving groups and each group has three members. Show that the number of groups is the same as the number of people at REU.

Problem 2. We have a table $m \times n$ with one real number in every cell. Prove that the sum of row sums is the same as the sum of column sums.

Identities with Binomial Coefficients

Definition. *Binomial Coefficient* $\binom{n}{k}$ represents the number of possibilities of selecting k objects out of n.

Problem 3. Prove
$$\binom{n}{k} = \binom{n}{n-k}.$$

Problem 4. Prove
$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Problem 5. Prove
$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Problem 6. Prove
$$\binom{n}{k} + \binom{n-1}{k} + \cdots + \binom{k}{k} = \binom{n+1}{k+1}.$$

Combinatorial Interpretation of Expressions

When using double counting it is important to be able to interpret number of possibilities you get combinatorially (then we just need to show that the expression on the other side of the equation gives the same result).

Problem 7. Interpret $n! = n \cdot (n-1) \cdot (n-2) \cdots 1$.

Problem 8. Interpret m^n.

Problem 9. Interpret
$$\binom{n}{m} \cdot 2^m.$$

Problem 10. Interpret
$$\frac{n(n-1)\cdots(n-k+1)}{k!}.$$

More Identities

Problem 11. (easy) Prove
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n,$$
$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

Problem 12. (Vandermonde's identity)
$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$

Problem 13. (medium) Prove
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n},$$
$$\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1},$$
$$\sum_{k=0}^{n} 2^k\binom{n}{k} = 3^n.$$

Problem 14. (hard) Prove
$$\sum_{i=0}^{m} \binom{n}{i}\binom{n-i}{m-i} = \binom{n}{m}2^m,$$
$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}.$$

More Problems

Problem 15. Prove that the sum of degrees of vertices in undirected graph is even.

Problem 16. Everyone at the party (there is at least one person) knows exactly seven boys and exactly ten girls (if A knows B then also B knows A). What is the minimum number of people at the party?

Problem 17. Let x_n be the number of strings of length n containing only letters A and B and not containing $ABABA$ and $BABAB$ as a consecutive substring. Let y_n be the number of strings of length n not containing five consecutive equal letters. Prove that $x_n = y_n$.

Definition. Tournament is a directed graph on n vertices with exactly one edge between any two vertices (the edge means who won against who). Let $\mathrm{out}(v)$ be the number of edges starting in vertex v and similarly $\mathrm{in}(v)$ number of edges ending in v.

Problem 18. (trivial) Prove that

$$\sum_{v \in V} \mathrm{in}(v) = \sum_{v \in V} \mathrm{out}(v).$$

Problem 19. Prove that in every tournament it is true that

$$\sum_{v \in V} (\mathrm{in}(v))^2 = \sum_{v \in V} (\mathrm{out}(v))^2.$$

Problem 20. There is a regular n-gon and every pair of vertices is connected by an edge. Count the number of intersecting pairs of edges (don't count if they touch in a vertex).

Problem 21. There are 2007 girls and 2007 boys in a school. Each of them is a member of at most 100 clubs. We know that for every pair of a girl and a boy there is a club that has both of them as members. Prove that there is a club with at least 11 girls and at least 11 boys.

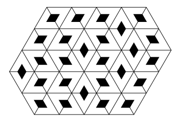
Hard Problems

Problem 22. In every cell of table $n \times n$ there is a number equal to the number of rectangles (or squares) containing this cell. Prove that the sum of these numbers is a second power of an integer.

Problem 23. There are three numbers a, b, c. Take all tables of nonnegative integers of dimension $a \times b$ where rows and columns are nonincreasing a all the numbers are at most c (left picture). On the other hand take a hexagon with internal angles

$120°$ and sides of lengths a, b, c, a, b, c and set of diamonds (right picture). Prove that the number of tables is the same as the number of possibilities how to fill the hexagon with diamonds.

2	2	1	1
2	2	0	0
1	1	0	0



Problem 24. Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called *full* if it satisfies the following condition: for each positive integer $k \geq 2$, if the number k appears in the sequence then so does the number $k - 1$, and moreover the first occurrence of $k - 1$ comes before the last occurrence of k. Show that there are $n!$ full sequences of length n.

Hints

1. Count number of pairs (person A, group containing person A) first time from the point of view of people and second time from the point of view of groups.

2. Say it is in both cases the sum of all the numbers.

3. You can pick few objects or discard the rest.

4. Do I pick the first object?

5. Don't forget about the leader.

6. What is the first object I pick?

7. Number of permutations.

8. Write $m^n = m \cdot m \cdots m$ and realize it is enought to repeat some choice n times.

9. First choose m people and then for each of them decide between two possibilities.

10. Choose k people out of n one by one. Realize how many times did you count every k-tuple of people.

11.

(i) Select any subset out of n people.
(ii) You want to select some normal and from them some special people.

12. Split $m+n$ people to two parts with m and n people.

13.

(i) The same as Vandermonde's identity.
(ii) Choose a captain and some more people.
(iii) First choose people not in the third group and decide if they are in the first or second group.

14.

(i) Remember how to interpret the right side.
(ii) What is the first person you don't choose?

15. Count number of pairs (vertex, edge from that vertex).

16. What is the number of edges between boys and girls? What is the minimum number of girls?

17. Invert every second letter.

18. Count number of edges.

19. Count all triplets of people such that one of them won both games. Realize that then one of them lost both games.

20. Take every quadraple of vertices.

21. Prove by contradiction. Count triplets (girl, boy, club containing both of them).

22. Count number of pairs (rectangle, cell in the rectangle). For this use three pointers to column (borders of rectangle and the column of the chosen cell) and the same for rows.

23. Draw the table on the left in 3D – the number is the number of cubes that sit on this cell.

24.

$$2,1,2,1,2,1,3,3 \Leftrightarrow 6,3,5,2,4,1,8,7$$