Advances on the EFL conjecture

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ADVANCES ON THE ERDŐS–FABER–LOVÁSZ CONJECTURE

David Romero and Abdón Sánchez-Arroyo

A hypergraph is linear if no two distinct edges intersect in more than one vertex. A well-known conjecture of Erdős, Faber, and Lovász states that if a linear hypergraph has n edges, each of size n, then there is a n-vertex colouring of the hypergraph such that each edge contains one vertex of each colour. Dating back to 1972, it is very surprising that this conjecture has not been settled in its full generality. Here we present some advances on it.

17.1 The original problem

Paul Erdős wrote in 1975 [8]:

> *Faber, Lovász, and I conjectured that if* $|A_k| = n$, $1 \le k \le n$, *and* $|A_k \cap A_j| \le 1$, *for* $k < j \le n$, *then one can colour the elements of the union* $\bigcup_{k=1}^{n} A_k$ *by* n *colours so that every set has elements of all the colours. It is very surprising that no progress has been made with this problem and I offer 50 pounds for a proof or disproof.*

Dating back to 1972, Erdős considered this combinatorial problem among his three favorites (see [8, 9, 10]). In this chapter, we present a survey of the work on the subject. In Section 17.2, we pose the problem in terms of graphs and hypergraphs. Section 17.3 is devoted to describe equivalent versions and some generalizations. Section 17.4 surveys the advances made towards the solution of this conjecture. Finally, some related problems are dealt with in Section 17.5.

17.2 The conjecture

A *hypergraph* $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ consists of a finite set $\mathcal{V}_\mathcal{H}$, the *vertices* of $\mathcal{H}$, and a finite family $\mathcal{E}_\mathcal{H}$ of non-empty subsets of $\mathcal{V}_\mathcal{H}$, the *edges* of $\mathcal{H}$; it is assumed that each vertex belongs to at least one edge. Thus a *graph* is a hypergraph in which every edge has at most two elements. A hypergraph $\mathcal{H}$ is *linear* if no two edges intersect in more than one vertex.

Let $\{1, \ldots, k\}$ be a set of k 'colours'. A (*proper*) *k-vertex colouring* of $\mathcal{H}$ is a map $\mathcal{V}_\mathcal{H} \mapsto \{1, \ldots, k\}$ such that no two vertices with the same colour belong to a common edge. The (vertex) *chromatic number* $\chi(\mathcal{H})$ of $\mathcal{H}$ is the least number of colours needed to colour properly $\mathcal{V}_\mathcal{H}$. Whereas this concept is traditionally used for graphs, for hypergraphs it is also known as the *strong chromatic number* (see Chapter 4 of [1]). However, we stick to our definition since it allows us to use it both in graphs and hypergraphs.

In the hypergraph setting the original Erdős–Faber–Lovász problem becomes one concerning vertex colourings.

Conjecture 17.1 ($\mathcal{EFL}$) *If $\mathcal{H}$ is a linear hypergraph consisting of n edges, each containing n vertices, then $\chi(\mathcal{H}) = n$.*

A *complete* graph is a graph in which every pair of vertices is an edge. The original $\mathcal{EFL}$-conjecture can also be stated in terms of graphs:

Conjecture 17.2 ($\mathcal{EFL}$: Graphical) *If a graph G is the union of n complete graphs $G_1, \ldots, G_n$ of size n, and $|V(G_i) \cap V(G_j)| \leq 1$, for $1 \leq i < j \leq n$, then $\chi(G) = n$.*

The Erdős–Faber–Lovász problem has a version in terms of edge colourings. A (*proper*) *k-edge colouring* of $\mathcal{H}$ is a map $\mathcal{E}_{\mathcal{H}} \mapsto \{1, \ldots, k\}$ such that no two edges with the same colour have a common vertex. The (*edge*) *chromatic index* $\chi_e(\mathcal{H})$ of $\mathcal{H}$ is the minimum number of colours needed to colour properly $\mathcal{E}_{\mathcal{H}}$. The *dual* of a hypergraph $\mathcal{H} = (\mathcal{V}_{\mathcal{H}}, \mathcal{E}_{\mathcal{H}})$ on p vertices $\mathcal{V}_{\mathcal{H}} = \{v_1, \ldots, v_p\}$, and n edges $\mathcal{E}_{\mathcal{H}} = \{E_1, \ldots, E_n\}$ is the hypergraph $\mathcal{H}^* := (\mathcal{V}^*_{\mathcal{H}}, \mathcal{E}^*_{\mathcal{H}})$ whose vertices $\mathcal{V}^*_{\mathcal{H}} = \{e_1, \ldots, e_n\}$ are the edges of $\mathcal{H}$, and whose p edges are defined by the sets: $V_i = \{e_j : v_i \in E_j \in \mathcal{H}\}$. It follows that $(\mathcal{H}^*)^*$ is isomorphic to $\mathcal{H}$.

The *bipartite graph representation* of a hypergraph $\mathcal{H} = (\mathcal{V}_{\mathcal{H}}, \mathcal{E}_{\mathcal{H}})$ is a bipartite graph $B(\mathcal{H}) = (V(B), E(B))$ with bipartition (X, Y) of $V(B)$ constructed as follows:

B.1 For each vertex $v \in \mathcal{V}_{\mathcal{H}}$ there is a vertex $x_v \in X$. Thus $|X| = |\mathcal{V}_{\mathcal{H}}|$.
B.2 For each edge $e \in \mathcal{E}_{\mathcal{H}}$ there is a vertex $y_e \in Y$. Thus $|Y| = |\mathcal{E}_{\mathcal{H}}|$.
B.3 The pair (x_v, y_e) is an edge in $B(\mathcal{H})$, if and only if $v \in e$.

We use this construction to explain properties about duality, vertex colourings, and edge colourings of hypergraphs. To start with we make three basic remarks.

Remark 17.3 The hypergraphs $\mathcal{H}$ and $\mathcal{H}^*$ have isomorphic bipartite graph representation (with their parts interchanged).

Remark 17.4 A vertex colouring of $\mathcal{H}$ corresponds to an edge colouring of $\mathcal{H}^*$ and vice versa. Namely, edge V_i of $\mathcal{H}^*$ receives the colour assigned to vertex v_i of $\mathcal{H}$; similarly, vertex e_j of $\mathcal{H}^*$ gets the colour assigned to edge E_j of $\mathcal{H}$.

Remark 17.5 *Duality preserves linearity.* Linearity of $\mathcal{H}$ is equivalent to the bipartite graph $B(\mathcal{H})$ containing no 4-cycles. This latter property is preserved when the two parts of $B(\mathcal{H})$ are interchanged, then $\mathcal{H}^*$ is also linear.

A hypergraph that satisfies the hypotheses of a conjecture will be called an *instance* of it. The *degree* $d_{\mathcal{H}}(v)$ of a vertex v in a hypergraph $\mathcal{H}$ is the number of edges containing it. A *loop* is an edge of cardinality one. We can prove that the $\mathcal{EFL}$-conjecture is equivalent to

Conjecture 17.6 ($\mathcal{EFL}$: Dual) *If $\mathcal{H}$ is a linear hypergraph with no repeated loops, then $\chi_e(\mathcal{H}) \leq |\mathcal{V}_\mathcal{H}|$.*

To prove the equivalence, consider an instance $\mathcal{H}$ of Conjecture 17.1 and let $\mathcal{H}^*$ be its dual. It follows from Remark 17.4 and Remark 17.5 that any n-edge colouring of $\mathcal{H}^*$ gives us a colouring of all vertices v of $\mathcal{H}$ with $d_\mathcal{H}(v) \geq 2$. However, such a colouring can be easily extended to the rest of $\mathcal{V}_\mathcal{H}$ so that each edge in $\mathcal{E}_\mathcal{H}$ has all n colours. On the other hand, let $\mathcal{H}$ be a instance of Conjecture 17.6. By Remark 17.5, $\mathcal{H}^*$ is linear and has $|\mathcal{V}_\mathcal{H}|$ edges each of size at most $|\mathcal{V}_\mathcal{H}|$. We add new vertices to each edge in $\mathcal{H}^*$ so that each edge has now size $|\mathcal{V}_\mathcal{H}|$. By Conjecture 17.1 we can find a vertex colouring with $|\mathcal{V}_\mathcal{H}|$ colours and this gives us the required edge colouring of $\mathcal{H}$.

The equivalence of Conjectures 17.1 and 17.6 was perhaps first noted by Hindman [17] and Seymour [27]. Besides, using computer search Hindman [17] was able to verify Conjecture 17.6 for 'small' hypergraphs, those having at most 10 vertices.

17.3 Related conjectures

In this section, we present some problems related to the $\mathcal{EFL}$-conjecture. First we deal with reformulations of the problem and then we present some generalizations of it.

17.3.1 *Equivalent versions*

We start this section with a version of the conjecture for hypergraphs of minimum degree at least two. The next version uses the fact that vertex colourings of hypergraphs are a special type of edge colourings of bipartite graphs. The third version uses the line graph of a hypergraph and a new parameter for general graphs. Finally, a translation of the original problem into algebraic language is presented.

Hypergraphs with $\delta(\mathcal{H}) \geq 2$

An easy reformulation of Conjecture 17.1 was used by Sánchez-Arroyo [26]. Let $\delta(\mathcal{H})$ denote the minimum degree of $\mathcal{H}$, that is, $\delta(\mathcal{H}) = \min_{v\in\mathcal{V}_\mathcal{H}}\{\delta_\mathcal{H}(v)\}$.

Conjecture 17.7 ([26]) *If a linear hypergraph $\mathcal{H}$ consists of n edges, each of size at most $n-1$, and $\delta(\mathcal{H}) \geq 2$, then $\chi(\mathcal{H}) \leq n$.*

To see that this conjecture implies Conjecture 17.1, consider an instance $\mathcal{H}$ of the latter. As $\mathcal{H}$ is linear and has n edges, each edge contains a vertex of degree one. By deleting from $\mathcal{H}$ every such vertex, we get a linear hypergraph with at most n edges, each of size at most $n-1$, and with minimum degree at least two. Observe that if we can properly colour the vertices of degree at least two (with n or fewer colours), then certainly we can colour all the vertices of $\mathcal{H}$ with n colours, which is the minimum number required.

Hybrid edge colourings

We now introduce a special type of edge-colourings of graphs and present its relationship with vertex colourings of hypergraphs. The motivation comes from attempting to solve the $\mathcal{EFL}$-conjecture using edge colourings of the bipartite graph representation of a hypergraph. The *star*, S_v, centred in a vertex v of a graph G, is the set of edges incident to v.

Consider a proper n-vertex colouring of a hypergraph $\mathcal{H}$. In the bipartite graph $B(\mathcal{H})$ such a colouring corresponds to an n-colouring of one of the parts of $B(\mathcal{H})$. Thus, the given n-vertex colouring of $\mathcal{V}_{\mathcal{H}}$ becomes an edge colouring of $B(\mathcal{H})$ as follows: If v is a coloured vertex in $B(\mathcal{H})$ then each edge in the star S_v receives the colour assigned to v, that is, S_v is monochromatic. On the other hand, given that the vertex colouring of $\mathcal{H}$ is proper, no two vertices in an edge E have the same colour. Thus, if v is not coloured in $B(\mathcal{H})$, then the star S_v is heterochromatic.

A set of vertices of a hypergraph is *independent* if no two belong to a common edge. Consider a simple graph $G = (V(G), E(G))$ and let A be an independent set of vertices of G. A *hybrid k-edge colouring* of G with respect to A is a map $E(G) \mapsto \{1, \ldots, k\}$ such that each star centred at a vertex in A is monochromatic, and each star centred at a vertex in $B = V(G) \backslash A$ is heterochromatic. The *hybrid chromatic index* $\chi_{\hat{e}}(G, A)$ of a simple graph G and an independent set A in G, is the minimum j such that G has a hybrid j-edge colouring.

We show how hybrid edge colourings of graphs determine proper vertex colourings of hypergraphs. Suppose that G is the bipartite graph representation of the hypergraph $\mathcal{H} = (\mathcal{V}_{\mathcal{H}}, \mathcal{E}_{\mathcal{H}})$, with bipartition (X, Y) of $V(G)$. Consider a hybrid edge colouring Ψ of graph G. Observe that, by definition, if $x_v \in X$ then $\Psi(x_v, y_1) = \Psi(x_v, y_2)$, for any pair of neighbours y_1, y_2 of x_v in G.

Define a colouring φ of the vertices of $\mathcal{H}$ by assigning to each vertex v in $\mathcal{V}_{\mathcal{H}}$ the colour $\Psi(x_v, y)$, that is, set $\varphi(v) = \Psi(x_v, y)$, where y is any neighbour of x_v. Observe that, by B.3, there is an edge (x_v, y_e) in G for every $v \in e$. Now, since Ψ is a hybrid edge colouring of G, no two edges incident to a vertex $y_e \in Y$ share colour. Hence, for any edge E of $\mathcal{H}$ no pair of vertices $u, w \in E$ are coloured the same by φ, and then φ is a proper vertex colouring of $\mathcal{V}_{\mathcal{H}}$. Thus, we have proved

Remark 17.8 Let k be a positive integer, and $\mathcal{H} = (V_{\mathcal{H}}, E_{\mathcal{H}})$ be a linear hypergraph with bipartite graph representation denoted by $B(\mathcal{H})$. Then, $\chi_{\hat{e}}(B(\mathcal{H}), X) \leq k$ if and only if $\chi(\mathcal{H}) \leq k$.

With this result it is clear that we can reformulate the $\mathcal{EFL}$-conjecture in terms of the hybrid chromatic index of bipartite graphs, as follows:

Conjecture 17.9 *Let $G = (V(G), E(G))$ be a a simple bipartite graph with bipartition (X, Y), then $\chi_{\hat{e}}(G, X) \leq |Y|$.*

Linear intersection number of graphs

In [22], Klein and Margraf used the line graph of a hypergraph to introduce a new parameter for general graphs, and then they provided a version of the conjecture in terms of this parameter.

A *partial linear space* is a pair $\mathcal{P} = (P, \mathcal{L})$ consisting of a set P of elements called *points* and a set $\mathcal{L}$ of subsets of P, called *lines*, satisfying the following axioms:

L.1 *Any two distinct points belong to at most one line.*
L.2 *Any line has at least two points.*

Observe that a partial linear space is just a linear hypergraph without loops. The *line graph* or *intersection graph* of a hypergraph $\mathcal{H}$ is the graph $L(\mathcal{H})$ whose vertices correspond to the edges of $\mathcal{H}$, with two vertices being adjacent if and only if their corresponding edges intersect.

Obviously, a vertex colouring of $L(\mathcal{H})$ corresponds to an edge colouring of $\mathcal{H}$. Hence, to determine the chromatic index of a hypergraph $\mathcal{P}$, it is enough to look at its intersection graph $L(\mathcal{P})$. However, it is not possible to recover the original hypergraph by looking only at its intersection graph. To cope with this problem, Klein and Margraf [22] introduced the linear intersection number of a graph.

Let $G = (V, E)$ be a simple graph without isolated vertices. The space $\mathcal{G} = (E, \{S_x; x \in V\})$ of G is a hypergraph whose line graph is G. Moreover, this hypergraph is almost a partial linear space. Any two lines of $\mathcal{G}$ intersect in at most one point. However, any vertex $x \in V$ with $d_G(x) = 1$ yields a line S_x of $\mathcal{G}$ with one point, and by L.2 such lines are not allowed in a partial linear space. But this is not a real problem, it suffices to add a new point to such line S_x. The partial linear space $\mathcal{G}^*$ obtained by this construction has the original graph as its intersection graph, namely, $L(\mathcal{G}^*) = G$.

In particular, each graph is the intersection graph of a linear hypergraph, and the set

$$N(G) = \{n \in \mathbb{N} : \mathcal{P} \text{ is a partial linear space on n points with } L(\mathcal{P}) = G\}$$

is not empty. The equality $L(\mathcal{P}) = G$ in this formula actually means isomorphism. Now the *linear intersection number* of G, $n(G)$, is defined as the smallest number of points of a partial linear space realizing G as its intersection graph. With this definition the $\mathcal{EFL}$-conjecture is equivalent to

Conjecture 17.10 ([22]) *If G is a graph, then $\chi(G) \leq n(G)$.*

For the equivalence, consider an instance of Conjecture 17.6 and let G be a graph. Observe that for every partial linear space $\mathcal{P} = (P, \mathcal{L})$, with

$L(\mathcal{P}) = G$, we have $|P| \geq \chi_e(\mathcal{P}) = \chi(L(\mathcal{P}))$; hence $n(G) \geq \chi(G)$. Conversely, let G be a graph with $\chi(G) \leq n(G)$. Then, for every partial linear space $\mathcal{P} = (P, \mathcal{L})$, with $L(\mathcal{P}) = G$, it follows that $|P| \geq n(G) \geq \chi(G) = \chi_e(\mathcal{P})$.

Maximal clones

Haddad and Tardif [16] translated the original conjecture into algebraic language. Given an instance $\mathcal{H}$ of Conjecture 17.1, we can define the relational structure $(V_{\mathcal{H}}, R_{\mathcal{H}})$, whose base set $V_{\mathcal{H}}$ is the same as that of $\mathcal{H}$, and $R_{\mathcal{H}}$ is the *n-ary relational structure* consisting of all the n-tuples $(x_1, \ldots, x_n)$ such that $\{x_1, \ldots, x_n\}$ is an edge of $\mathcal{H}$. Thus, if $\mathcal{H}$ has n edges, then $R_{\mathcal{H}}$ consists of the $n \times n!$ n-tuples obtained by linearly ordering each edge of $\mathcal{H}$ in all possible ways.

An n-ary relational structure (V, R) is called *areflexive* if for every $(x_1, \ldots, x_n) \in R$, the elements $x_1, \ldots, x_n$ are all distinct, and *totally symmetric* if for every $(x_1, \ldots, x_n) \in R$ and every permutation π of $\{1, \ldots, n\}$, we have $(x_{\pi(1)}, \ldots, x_{\pi(n)}) \in R$. A *homomorphism* between two n-ary relational structures (V, R) and (V_0, R_0) is a map ϕ from V to V_0 such that $(\phi(x_1), \ldots, \phi(x_n)) \in R_0$ for all $(x_1, \ldots, x_n) \in R$. A map ϕ from a subset Y of V to V_0 is called a *partial homomorphism* if $(\phi(x_1), \ldots, \phi(x_n)) \in R_0$ for all $x_1, \ldots, x_n \in Y$ such that $(x_1, \ldots, x_n) \in R$.

Given an integer m, the m-th power $(V, R)^m$ of an n-ary relational structure (V, R) is the n-ary relational structure (V^m, R_0), where

$$R_0 = \{(X_1, \ldots, X_n) \in (V^m)^n : \ (p_i(X_1), \ldots, p_i(X_n)) \in R, \ i = 1, \ldots, m\},$$

and p_i is the projection given by $p_i(a_1, \ldots, a_m) = a_i$. A partial function from a power of V to V is called a *partial operation* on V. Let $Po(V, R)$ denote the set of partial operations on V which are partial homomorphisms from some power of (V, R) to (V, R).

A set of partial operations containing all projections and closed under composition is called a *partial clone*. A partial clone is called *maximal* if it is not properly contained in any other partial clone, apart from the set of all partial operations on V. Haddad and Rosenberg characterized maximal partial clones on a finite set by means of

Theorem 17.11 ([15]) *Let (V, R) be an n-ary areflexive, totally symmetric relational structure. Then $Po(V, R)$ is a maximal partial clone if and only if (V, R) admits an n-colouring.*

Here, an n-colouring of (V, R) is just an n-vertex colouring of the hypergraph whose edges are the sets $\{x_1, \ldots, x_n\}$ such that $(x_1, \ldots, x_n) \in R$. Haddad and Tardif deduced from Theorem 17.11 the following:

Corollary 17.12 ([16]) *Let $\mathcal{H}$ be an instance of Conjecture 17.1. Then $\mathcal{H}$ admits an n-colouring if and only if $Po(V_{\mathcal{H}}, R_{\mathcal{H}})$ is a maximal partial clone.*

From this corollary, it follows that the $\mathcal{EFL}$-conjecture is equivalent to

Conjecture 17.13 ([16]) *For every instance $\mathcal{H}$ of the $\mathcal{EFL}$-conjecture, the relational structure $(V_{\mathcal{H}}, R_{\mathcal{H}})$ determines a maximal partial clone.*

17.3.2 *Generalizations*

Here we present two problems that generalise the $\mathcal{EFL}$-conjecture, namely, the Linear Hypergraph Conjecture and the Alon–Saks–Seymour conjecture.

The linear hypergraph conjecture

For simple graphs Conjecture 17.6 is contained in Vizing's Theorem.

Theorem 17.14 ([30]) *If $G = (V(G), E(G))$ is a simple graph of maximum degree $\Delta(G)$, then $\chi_e(G) \leq \Delta(G) + 1$.*

Let k be a positive integer. The *strict k-section* $\mathcal{H}_{<k>}$ on a hypergraph $\mathcal{H}$ is defined by all subsets $F \subseteq \mathcal{V}_{\mathcal{H}}$ such that $1 \leq |F| \leq k$, and $F \subset e$, for some $e \in \mathcal{E}_{\mathcal{H}}$. Observe that $\mathcal{H}_{<2>}$ is a graph with a loop attached to each vertex. Thus, if $\mathcal{H} = (\mathcal{V}_{\mathcal{H}}, \mathcal{E}_{\mathcal{H}})$ is a simple graph then $\Delta(\mathcal{H}_{<2>}) = \Delta(\mathcal{H}) + 1$, and so Vizing's theorem is a special case of the following:

Conjecture 17.15 (Linear) *If $\mathcal{H}$ is a linear hypergraph, then $\chi_e(\mathcal{H}) \leq \Delta(\mathcal{H}_{<2>})$.*

This conjecture was stated independently by Meyniel (unpublished), Berge [2], and Fűredi [13]. Observe that the line graph $L(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the strict 2-section of $\mathcal{H}^*$ with all loops removed, so $\Delta(\mathcal{H}^*_{<2>}) = \Delta(L(\mathcal{H})) + 1$. From this, they found that the Linear Hypergraph Conjecture is equivalent to

Conjecture 17.16 (Linear: Dual) *Let $\mathcal{H}$ be a linear hypergraph with no edge containing more than one vertex of degree one. Then $\chi(\mathcal{H}) \leq \Delta(L(\mathcal{H})) + 1$.*

It is clear that the dual version of the Linear Hypergraph Conjecture contains the $\mathcal{EFL}$-conjecture.

The Alon–Saks–Seymour conjecture

Kahn [20] reported that Alon, Saks, and Seymour proposed

Conjecture 17.17 *If the edge set of a graph G is the disjoint union of the edge sets of n complete bipartite graphs, then $\chi(G) \leq n + 1$.*

It can be seen that this is a generalization of the following theorem by Graham and Pollak.

Theorem 17.18 ([14]) *The edge set of the complete graph K_n cannot be partitioned into the edge set of fewer than $n - 1$ complete bipartite graphs.*

There is a short algebraic proof of this result due to Tverberg [29], but no combinatorial proof seems to be known. The Alon–Saks–Seymour conjecture is a bipartite analogue of Conjecture 17.2. To see this, replace in the former 'complete bipartite graphs' with 'complete graphs on n vertices', and '$\chi(G) \leq n+1$' with '$\chi(G) \leq n$', to obtain a formulation of the $\mathcal{EFL}$-conjecture.

17.4 Results on the $\mathcal{EFL}$-conjecture

Advances on the $\mathcal{EFL}$-conjecture are scarce, and surprisingly most of them have been originally obtained for the edge colouring version, Conjecture 17.6. In this section, we consider this dual version of the $\mathcal{EFL}$-conjecture unless otherwise stated.

A set of edges of $\mathcal{H}$ is a *matching* if they are pairwise disjoint. Observe that, apart from the trivial cases, when $\mathcal{H}$ has no edges or when it has one vertex, Conjecture 17.6 is sharp in each of the following cases:

E.1 $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$, where $\mathcal{E}_\mathcal{H} = \{\{1\}, \{1,2\}, \{1,3\} \ldots, \{1, |\mathcal{V}_\mathcal{H}|\}$.

E.2 $\mathcal{H}$ is a degenerate projective plane, that is, $\mathcal{V}_\mathcal{H} = \{0, 1, \ldots, n-1\}$, and $\mathcal{E}_\mathcal{H} = \{\{0,1\}, \ldots, \{0, n-1\}, \{1, 2, \ldots, n-1\}\}$.

E.3 $\mathcal{H}$ is a projective plane, where $\mathcal{V}_\mathcal{H}$ is the set of points and $\mathcal{E}_\mathcal{H}$ is the set of lines (see Fig. 17.1b).

E.4 $\mathcal{H}$ is a complete graph on n vertices, n odd, $n \geq 5$.

Remark 17.19 Some 'minor' modifications of the hypergraphs described in E.4 also give equality in Conjecture 17.6. By minor modifications we mean the deletion of some edges from the complete odd graph K_{2p+1}. Because $K_{2p+1} - e$ has $2p^2 + p - 1$ edges and each matching of it has at most p edges, we get $\chi_e(K_{2p+1} - e) \geq 2p + 1$. For example, $\chi_e(K_5 - e) = 5$, because $K_5 - e$ has nine edges and we need five colours to colour its edges.

We start with a relatively old theorem by de Bruijn and Erdős [4] concerning the problem for *intersecting hypergraphs*, that is, those in which any two edges intersect.

Theorem 17.20 ([4]) *Let $\mathcal{H}$ be an intersecting linear hypergraph. Then $|\mathcal{E}_\mathcal{H}| \leq |\mathcal{V}_\mathcal{H}|$, with equality in the cases E.1, E.2, and E.3.*

Seymour made an important step by extending this theorem in the following way.

Theorem 17.21 ([27]) *Every linear hypergraph $\mathcal{H} = (\mathcal{V}_\mathcal{H}, \mathcal{E}_\mathcal{H})$ has a matching of size at least $|\mathcal{E}_\mathcal{H}|/|\mathcal{V}_\mathcal{H}|$.*

The proof uses such a sophisticated argument that we do not include it here. Moreover, in [27] there is a proof that equality holds in this theorem for the hypergraphs described in E.1 to E.4.

Colbourn and Colbourn [7] proved Conjecture 17.6 for cyclic Steiner systems. A *Steiner system* $S(2, k, n)$ is a pair $(V, \mathcal{B})$; V is a set of n elements and $\mathcal{B}$ is a collection of subsets of V each of cardinality k called *blocks*. Each pair of elements of V appears in precisely one block. Thus, a Steiner system is just a *uniform* linear hypergraph, that is, all the edges have the same cardinality. A Steiner system $S(2,k,n)$ is *cyclic* if its element set is $\{0,1,\dots,n-1\}$ and the mapping $i \to i+1 \mod n$ is an automorphism. Using Brooks' theorem Colbourn and Colbourn were able to prove

Theorem 17.22 ([7]) *If $S(2,k,n)$ is a cyclic Steiner system, then $\chi_e(S(2,k,n)) \leq n$.*

Chang and Lawler established an upper bound for the chromatic index of a linear hypergraph in terms of its number of vertices.

Theorem 17.23 ([5]) *If $\mathcal{H}$ is a linear hypergraph, then $\chi_e(\mathcal{H}) \leq \frac{3}{2}|\mathcal{V}_\mathcal{H}| - 2$.*

This bound was obtained by using a greedy procedure to colour the edges of size at most three and a counting argument for the uncoloured part of the hypergraph. It has some similarity with Shannon's inequality for the chromatic index of graphs [28].

Later, Kahn [19] proved that Conjecture 17.6 is asymptotically correct, based upon a probabilistic technique developed by Rödl [25] to prove the existence of a large matching in a hypergraph with certain regularity properties. One generalization of Rödl's result is due to Pippenger and Spencer [24]. Kahn generalized significantly the main result in [24] dealing with the chromatic index of uniform hypergraphs, and then he used it to prove

Theorem 17.24 ([19]) *If $\mathcal{H}$ is a linear hypergraph, then $\chi_e(\mathcal{H}) = n + o(n)$.*

Kahn and Seymour [21] proved a relaxation of the conjecture that had been proposed in [27], namely, the *fractional* version of the dual $\mathcal{EFL}$-conjecture. Let $\Re^+$ be the set of non-negative real numbers and let $\mathcal{M}_\mathcal{H}$ be the set of matchings of a linear hypergraph $\mathcal{H}$.

Theorem 17.25 (Fractional $\mathcal{EFL}^*$ [21]) *Let $\mathcal{H}$ be a linear hypergraph. There is a function $f : \mathcal{M}_\mathcal{H} \mapsto \Re^+$ satisfying:*

1. $\sum_{M \in \mathcal{M}_\mathcal{H}} f(M) \leq |\mathcal{V}_\mathcal{H}|$, *and*
2. *for every edge $e \in \mathcal{E}_\mathcal{H}$, $\sum_{M \in \mathcal{M}_e} f(M) \geq 1$, where $\mathcal{M}_e$ is the set of matchings in $\mathcal{M}_\mathcal{H}$ containing the edge e.*

This result was proved using a lemma due to Motzkin:

Lemma 17.26 ([23]) *Let G be a simple bipartite graph with bipartition (X, Y), such that $X \neq \emptyset$ and no vertex in X is adjacent to every vertex in Y. Then, there exist $x \in X$ and $y \in Y$ such that x and y are not adjacent and $|Y| d_G(y) \leq |X| d_G(x)$.*

It is easy to see that the dual $\mathcal{EFL}$-conjecture is equivalent to the assertion that the function f in the fractional $\mathcal{EFL}^*$ may be chosen 0,1-valued. By Linear Programming duality (see [6, Chapter 5]) the fractional $\mathcal{EFL}^*$ is equivalent to

FIG. 17.1. The beauty of Fano.

Theorem 17.27 ([21]) *Let $\mathcal{H}$ be a linear hypergraph, and let $p : \mathcal{E}_{\mathcal{H}} \mapsto \Re^+$ satisfy $\sum_{e\in M} p(e) \leq 1$ for every matching M. Then $\sum_{e\in\mathcal{E}_{\mathcal{H}}} p(e) \leq |\mathcal{V}_{\mathcal{H}}|$.*

Quite recently, Sánchez-Arroyo [26] solved the conjecture for dense, linear hypergraphs, with the novelty that his approach was through the vertex colouring version, as stated in Conjecture 17.7.

Call a hypergraph $\mathcal{H}$ *dense* if its minimum degree, $\delta(\mathcal{H})$, is greater than $\sqrt{|\mathcal{E}_{\mathcal{H}}|}$. Among the possible examples of dense linear hypergraphs we have all finite projective planes, a conspicuous one being Fano's configuration. Figure 17.1(a) shows a linear, intersecting hypergraph $\mathcal{H}$ with seven edges (six straight lines and one circle), each with seven vertices, for a total of 35 vertices. Removing from $\mathcal{H}$ the vertices with degree one we obtain a dense hypergraph (Fano) as shown in Figure 17.1(b); it is regular of degree three.

One can prove Conjecture 17.7 for dense hypergraphs using the greedy algorithm to colour the vertices in descending order according to the degree sequence of the hypergraph, as used by Welsh and Powell in [31].

Theorem 17.28 ([26]) *Let $\mathcal{H}$ be a dense linear hypergraph consisting of n edges, each of size at most $n-1$, with $\delta(\mathcal{H}) \geq 2$. Then $\chi(\mathcal{H}) \leq n$.*

17.5 Related problems

In this last section, we state the results known to us for Conjecture 17.15, and present some new problems.

17.5.1 *On the linear hypergraph conjecture*

From a counting argument and Brooks' theorem we get

Remark 17.29 Let $\mathcal{H}$ be a linear r-uniform hypergraph, with maximum degree $\Delta(\mathcal{H}) < r$. Then $\chi_e(\mathcal{H}) \leq \Delta(\mathcal{H}_{<2>})$, equality holding if $\mathcal{H}$ is a projective plane or an odd cycle.

An interesting application of edge colourings of graphs was described by Berge and Hilton [3].

Theorem 17.30 ([3]) *Let $\mathcal{H}$ be a linear hypergraph with no repeated loops, in which the set of edges of cardinality greater than two is a matching. Then $\chi_e(\mathcal{H}) \leq \Delta(\mathcal{H}_{<2>})$.*

The idea of the proof is to choose an appropriate matching, $M = M_1 \cup M_2$, of $\mathcal{H}$, where M_2 is the matching formed by the edges of cardinality greater than two, and M_1 is a maximal matching of the remaining hypergraph after removing all edges touching M_2 and the vertices contained in edges of M_2. Now it is easy to colour the graph obtained from $\mathcal{H}$ by removing the edges of M_2 using the fact, due to Fournier [12], that *if G is a simple graph and the subgraph induced by the vertices of maximum degree $\Delta(G)$ is a forest then $\chi_e(G) = \Delta(G)$* (for a proof of this result see [11]).

Following a suggestion by Berge [2], both Füredi [13], and Kahn, Lovász, and Seymour, as reported in [21], proved the Linear Hypergraph Conjecture for intersecting hypergraphs. The proof is a consequence of Lemma 17.26. Let $\Delta_0(\mathcal{H})$ denote the maximum size of a pairwise intersecting set of edges in $\mathcal{H}$.

Theorem 17.31 ([21]) *If $\mathcal{H}$ is a linear hypergraph, then $\Delta_0(\mathcal{H}) \leq \Delta(H_{<2>})$.*

The proof goes as follows: First, the bipartite graph representing the hypergraph $(N_x, \mathcal{E}_{\mathcal{H}})$ is constructed, where $x \in \mathcal{V}_{\mathcal{H}}$ is a vertex of maximum degree, and N_x is its neighbourhood in the graph $\mathcal{H}_{<2>}$. Then, after disposing of some easy cases, it is possible to apply Lemma 17.26 to obtain the result.

A *partial hypergraph* of a hypergraph $\mathcal{H}$ is the hypergraph $\mathcal{H}'$ that arises by selecting a subset of edges from $\mathcal{H}$, the union of those edges being the set of vertices of $\mathcal{H}'$. Jackson, Sethuraman, and Whitehead obtained

Theorem 17.32 ([18]) *Let $\mathcal{H}$ be a loopless linear hypergraph and let $\mathcal{H}'$ be the partial hypergraph of $\mathcal{H}$ determined by the edges of size at least three. If $\mathcal{H}'$ satisfies $\chi_e(\mathcal{H}') = \Delta_0(\mathcal{H}') = \Delta(\mathcal{H}')$ and $\Delta(\mathcal{H}') \leq 3$, then $\chi_e(\mathcal{H}) \leq |\mathcal{V}_{\mathcal{H}}|$.*

17.5.2 *New problems*

Problem 17.33 *How easy is it to determine $\chi_{\hat{e}}(G, A)$ for a simple graph G and independent set A?*

Problem 17.34 *Find bounds for $\chi_{\hat{e}}(G, A)$.*

Problem 17.35 *Consider a simple bipartite graph $G = (V(G), E(G))$ with bipartition (A, B). What conditions must be satisfied by G in order to have a hybrid edge colouring with respect to A using at most $|B|$ colours?*

Obviously, this latter problem is posed when $|A| > |B|$, as otherwise we trivially obtain a hybrid $|A|$-edge colouring of G by assigning colour i to the edges of the star centred at vertex x_i of A, for $i = 1, \ldots, |A|$; since G is simple each star centred at a vertex of B is heterochromatic.

To close this section we pose a counting problem related to the topics so far seen. Let $\omega(n)$ be the number of linear, intersecting hypergraphs with n edges, each edge of size n. By hand, one can find that $\omega(2) = 1$, $\omega(3) = 2$, $\omega(4) = 3$, $\omega(5) = 5$, $\omega(6) = 10$, and $\omega(7) = 27$. However, going further seems difficult.

Problem 17.36 *Determine $\omega(n)$ for all n.*

References

[1] C. Berge, *Hypergraphs, Combinatorics of Finite Sets*, North Holland, Amsterdam (1989).

[2] C. Berge, On the chromatic index of a linear hypergraph and the Chvátal conjecture, Proc. Symposium, June 10–14 1985, *Ann. N.Y. Acad. Sci. Proc.* **555** (1989).

[3] C. Berge and A. J. W. Hilton, On two conjectures about edge-colouring hypergraphs, *Congressus Numerantium* **70** (1990) 99–104.

[4] N.G. de Bruijn and P. Erdős, A combinatorial problem, *Indagationes Math.* **8** (1946) 461–467.

[5] W. I. Chang and E. Lawler, Edge colorings of hypergraphs and a conjecture of Erdős, Faber, and Lovász, *Combinatorica* **8** (1988) 293–295.

[6] V. Chvátal, *Linear Programming*, W. H. Freeman and Company, New York, (1983).

[7] C. J. Colbourn and M. J. Colbourn, The chromatic index of cyclic Steiner 2-designs, *Internat. J. Math. & Math Sci.* **5/4** (1982) 823–825.

[8] P. Erdős, Problems and results in graph theory and combinatorial analysis. *Proc. 5th British Combinatorial Conference 1975* (C. St. J. A. Nash-Williams and J. Sheehan, eds.), *Congressus Numerantium XV. Utilitas Mathematica*, Winnipeg (1976) pp. 169–192.

[9] P. Erdős, On the combinatorial problems which I would most like to see solved, *Combinatorica* **1** (1981) 313–318.

[10] P. Erdős, Selected problems, in *Progress in Graph Theory* (J. A. Bondy and U. S. R. Murty, eds.), Academic Press (1984) pp. 528–531.

[11] S. H. Fiorini and R. J. Wilson, *Edge colourings of graphs. Research Notes in Mathematics* **16** Pitman, London, (1977).

[12] J. C. Fournier, Colorations des arêtes d'un graphe, *Cahiers du CERO* **15** (1973) 311–314.

[13] Z. Füredi, The chromatic index of simple hypergraphs, *Graph and Combinatorics* **2** (1986) 89–92.

[14] R. L. Graham and H. P. Pollak, On embedding graphs in squashed cubes, in (Y. Alavi, D. R. Lick, and A. T. White, eds.) Graph Theory with Applications, *Lecture Notes on Mathematics* **303**, Springer-Verlag (1972) pp. 99–110.

[15] L. Haddad and I. G. Rosenberg, Maximal partial clones determined by areflexive relations, *Discrete Appl. Math.* **24** (1989) 133–143.

[16] L. Haddad and C. Tardif, A clone-theoretic formulation of the Erdős-Faber-Lovász Conjecture, *Discussiones Mathematicae Graph Theory* **24/3** (2004) 545–549.

[17] N. Hindman, On a conjecture of Erdős, Faber, and Lovász about n-colorings, *Canadian J. Math.* **33** (1981) 563–570.

[18] B. Jackson, G. Sethuraman, and C. Whitehead, A note on the Erdős-Faber-Lovász conjecture, Preprint (2003).

[19] J. Kahn, Coloring nearly-disjoint hypergraphs with $n + o(n)$ colors, *Journal Combin. Theory Ser. A* **59** (1992) 31–39.

[20] J. Kahn, Recent results on some not-so-recent hypergraph matching and covering problems, in *Extremal problems for finite sets*, (P. Frankl, Z. Füredi, G. O. H. Katona, and D. Miklós, eds.), *Bolyai Society Mathematical Studies* **3** Janos Bolyai Mathematical Society (1994).

[21] J. Kahn and P. D. Seymour, A fractional version of the Erdős-Faber-Lovász conjecture, *Combinatorica* **12** (1992) 155–160.

[22] H. Klein and M. Margraf, On the linear intersection number of graphs, Preprint (2003).

[23] T. S. Motzkin, The lines and planes connecting the points of a finite set, *Trans. Amer. Math. Soc.* **70** (1951) 451–469.

[24] N. Pippenger and J. Spencer, Asymptotic behavior of the chromatic index for hypergraphs, *J. Combin. Theory Ser. A* **51** (1989) 24–42.

[25] V. Rödl, On a packing and covering problem, *Europ. J. Combinatorics* **5** (1985) 69–78.

[26] A. Sánchez-Arroyo, The Erdős–Faber–Lovász conjecture for dense hypergraphs, *Proceedings of the 20th British Combinatorial Conference, Durham U.K.* (2005) accepted.

[27] P. D. Seymour, Packing nearly-disjoint sets, *Combinatorica* **2** (1982) 91–97.

[28] C. E. Shannon, A theorem on coloring the lines of a network, *J. Math. Phys.* **28** (1949) 148–151.

[29] H. Tverberg, On the decomposition of K_n into complete bipartite graphs, *J. Graph Theory* **6** (1982) 493–494.

[30] V. G. Vizing, On an estimate of the chromatic class of a p-graph (in Russian), *Diskret. Analiz* **3** (1964) 25–30.

[31] D. J. A. Welsh and M. B. Powell, An upper bound for the chromatic number of a graph and its application to timetabling problems, *The Computer Journal* **10/1** (1967) 85–86.