THE CHROMATIC POLYNOMIAL OF A GRAPH

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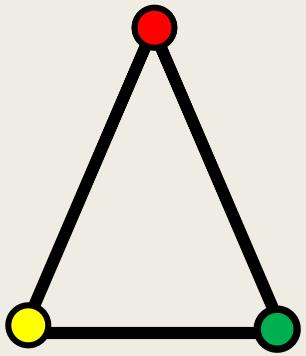
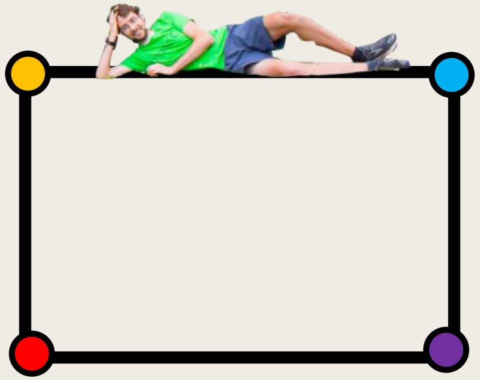
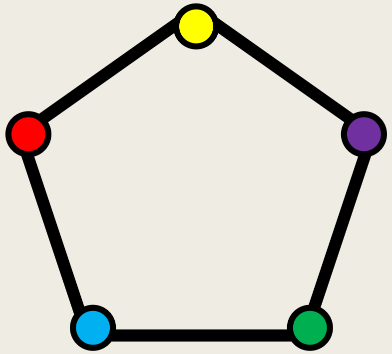
Project on Pure Math: Geometry and Combinatorics of Matroids

The Chromatic Polynomial

$$\chi_G(q)$$

- Proper Coloring:
  - *Two vertex on the same edge must have different colors*
- Definition of Chromatic Polynomial:
  - *The number of ways to color the vertices of a graph, G, using q colors so that the endpoints of every edge have different colors.*

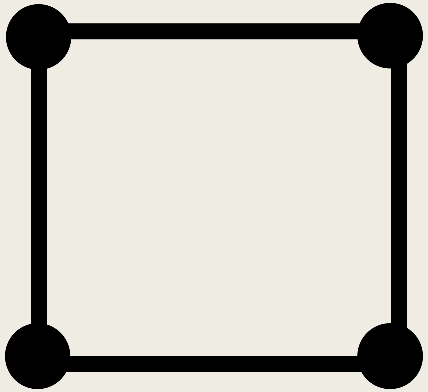
Deletion-Contraction Relation

- $$\chi_{G}(q) = \chi_{G\backslash \mathrm{e}}(q) - \chi_{G/e}(q)$$

G\e – deletion of an edge, e, from the graph, G

G/e- contraction of an edge from the graph

Example of C_4 Graph



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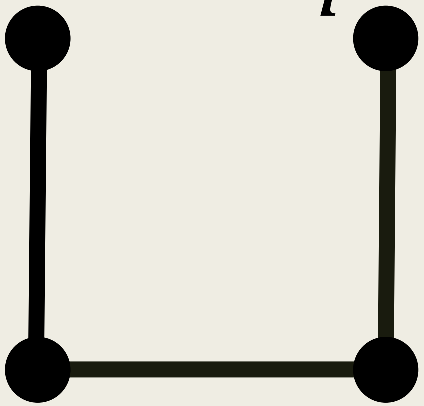

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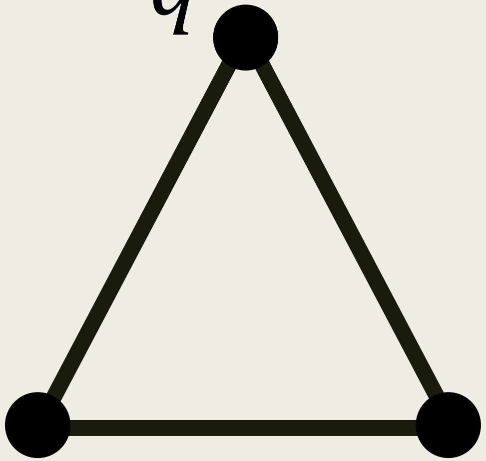
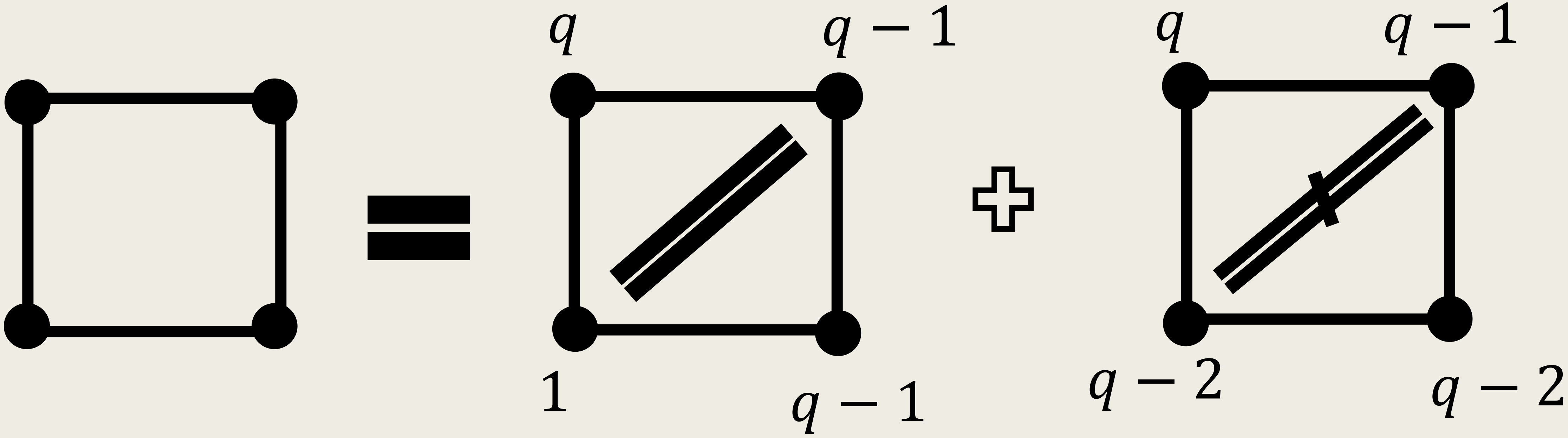


$$= \quad q(q-1)^3 - q(q-1)(q-2)$$

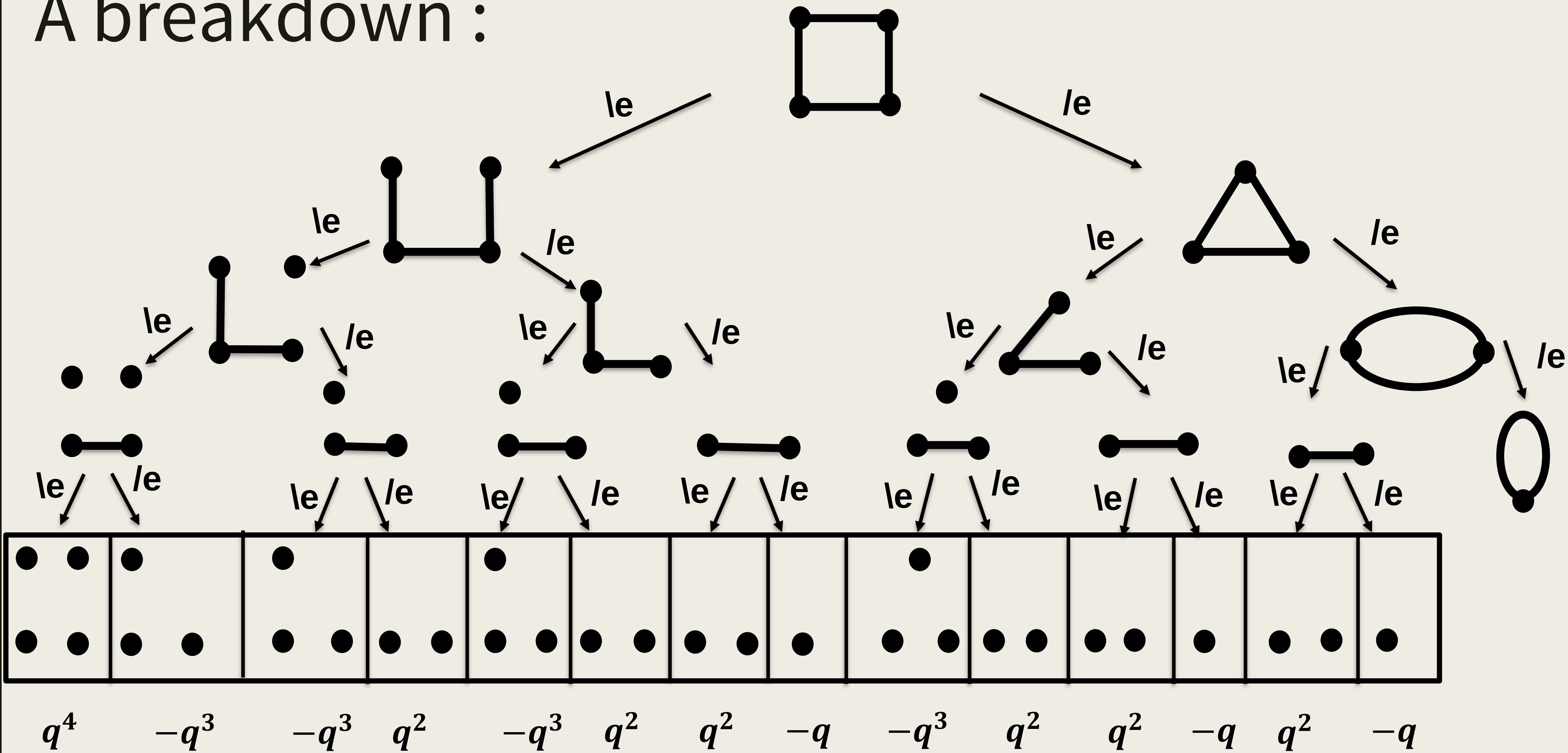
$$= \quad q^4 - 4q^3 + 6q^2 - 3q$$



$$= q(q-1)^2 + q(q-1)(q-2)^2$$

$$= q^4 - 4q^3 + 6q^2 - 3q$$

A breakdown :



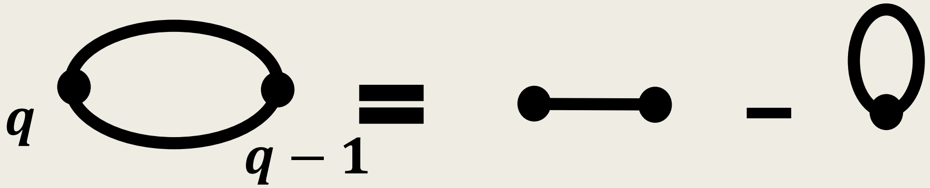
$$q^4 - 4q^3 + 6q^2 - 3q$$

Chromatic Polynomial Of Cycles:

n = 1

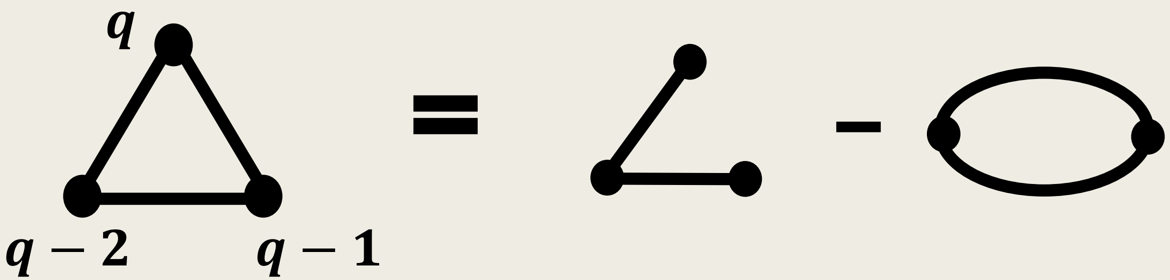
$$\chi(q) = 0$$

n = 2



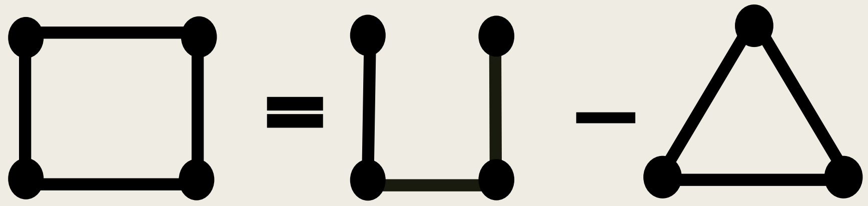
$$\chi(q) = q(q-1)$$

n = 3



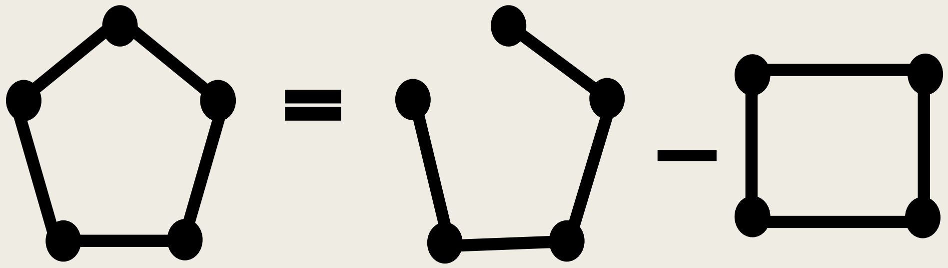
$$\chi(q) = q(q-1)(q-2)$$

n = 4



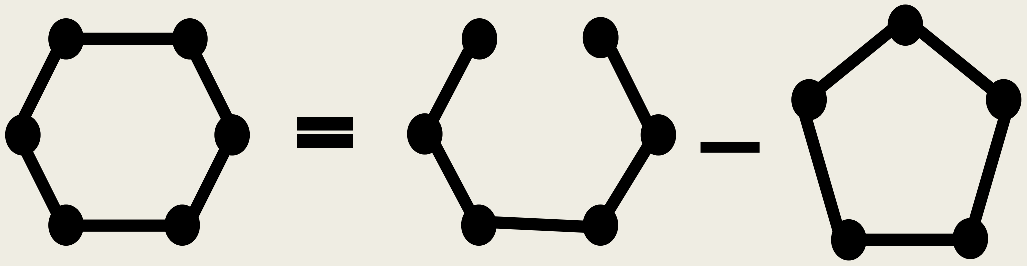
$$\chi(q) = q(q-1)^2 + q(q-1)(q-2)^2$$

n = 5



$$\chi(q) = q(q-1)^4 - \left(q(q-1)^2 + \left(q(q-1)(q-2)^2\right)\right)$$

n = 6



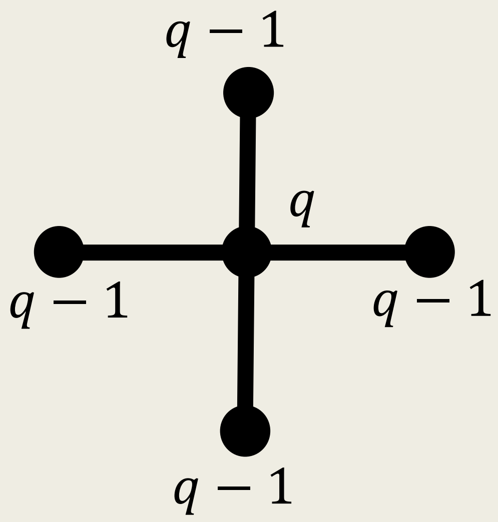
$$\chi(q) = q(q-1)^5 - \left(q(q-1)^4 - \left(q(q-1)^2 + \left(q(q-1)(q-2)^2\right)\right)\right)$$

General Solution for a Cycle:

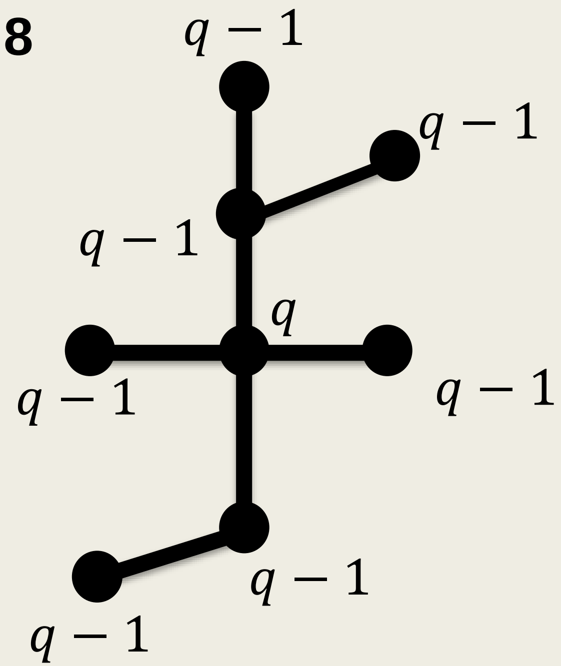
- $\chi_{C_n}(q) = \chi_{C_n \backslash \mathrm{e}} - \chi_{C_{n-1}}(q)$

Trees:

n = 5



n = 8



$$\chi(q) = q(q-1)^4$$

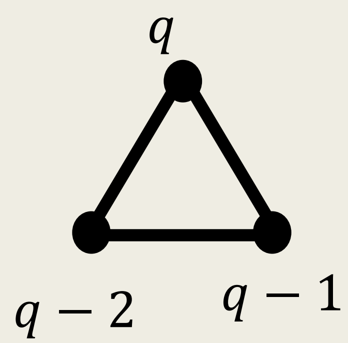
$$\chi(q) = q(q-1)^7$$

General Solution for Trees:

$$\chi(q) = q(q-1)^{n-1}$$

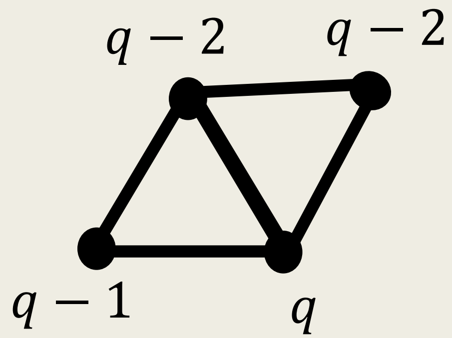
Open Wheels:

$n = 3$



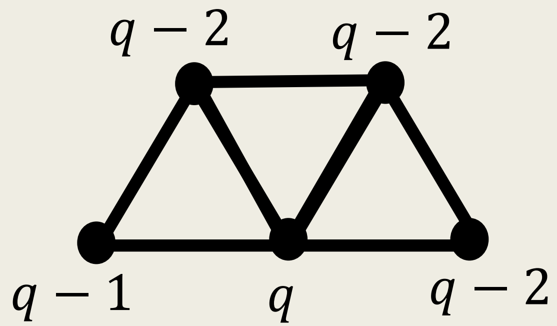
$q(q-1)(q-2)$

$n = 4$



$q(q-1)(q-2)^2 \longrightarrow \chi(q) = q(q-1)(q-2)^{n-2}$

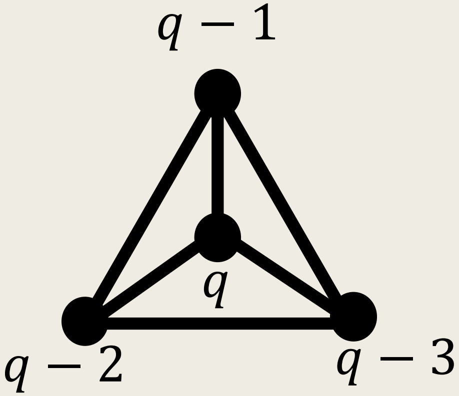
$n = 5$



$q(q-1)(q-2)^3$

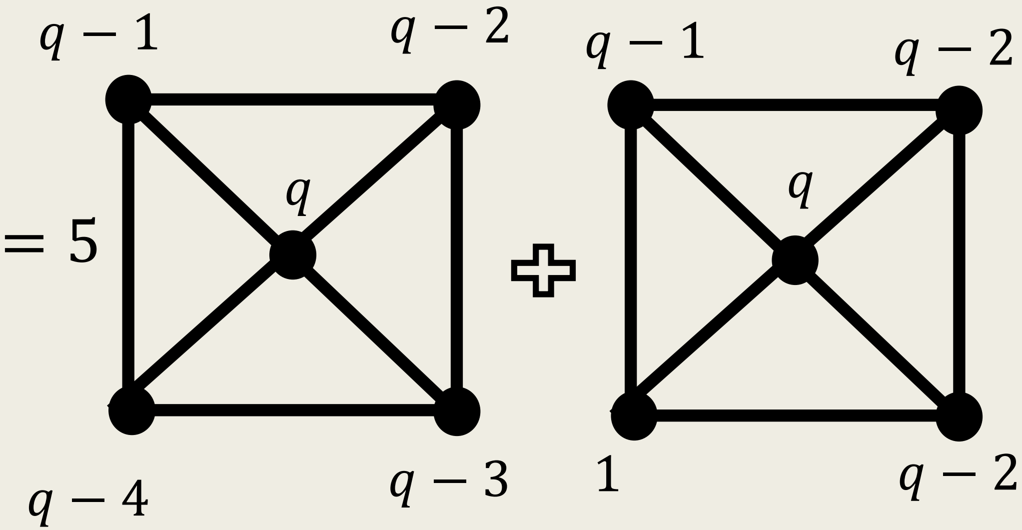
Wheels:

$n = 4$



$$q(q-1)(q-2)(q-3)$$

$n = 5$



$$q(q-1)(q-2)(q-3)^2 + q(q-1)(q-2)^2$$

General Solution: $q\left[(q-2)^{n-1} + (-1)^{n-1}(q-2)\right]$

Current Work:

- Working on finding the Chromatic Polynomial of a Matroid! That is a little more complicated because of the rules that apply to matroids.
- Also I’m working on the chromatic polynomial of 3-dimensional graphs.

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Citations:

- Huh, J. (2018). Combinatorial Applications Of The Hodge-Riemann Relations. April 2018.
- Hillman, H. (n.d.). Matroid Theory. [online] Available at: https://www.whitman.edu/Documents/Academics/Mathematics/hillman.pdf [Accessed 1 Jul. 2018].
- Oxley, J. (1993). *Matroid Theory*. New York: Oxford University Press.