Interpolation of holomorphic functions

For RUMA math mingle, 23 April, 2021.

Definition

Let $U \subseteq \mathbb{C}$ be open and connected. We say that $f : U \to \mathbb{C}$ is **holomorphic** if for all $z \in U$, the following limit exists:

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

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In particular, we restrict to $\mathrm{End}(\mathbb{D}) := \mathcal{O}(\mathbb{D}, \mathbb{D})$, the space of holomorphic functions mapping the unit disk into itself.

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Problem

Given **initial data** $z_1, z_2, \ldots, z_n \in \mathbb{D}$, *and* **target data** $w_1, w_2, \ldots, w_n \in \mathbb{D}$, *does there exist a holomorphic function* $f \in \mathrm{End}(\mathbb{D})$ *so that* $f(z_i) = w_i$ *for all* $1 \leq i \leq n$*?*

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Note that this problem is trivial in the case of $\mathcal{O}(\mathbb{D})$ as $n+1$ many points in $\mathbb{C}$ determine a unique polynomial of degree at most n (see the Vandermonde matrix).

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Thus, we work with $\mathbb{D}$ as otherwise the problem is too easy.

Example

Initial data: $0, 2/5$, and target data: $1/2, 3/4$.

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Such an f exists by checking:

$$f(z) = \frac{z + 1/2}{1 + z/2}.$$

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Such an f does not exist.

Theorem

(Schwarz-Pick) Fix $f \in \operatorname{End}(\mathbb{D})$. *For all* $z, w \in \mathbb{D}$,

$$\left|\frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)}\right| \leq \left|\frac{z - w}{1 - \overline{w}z}\right|.$$

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We see that,

$$\left|\frac{1/2 - 3/4}{1 - (1/2) \cdot (3/4)}\right| = 2/5 \text{ and } \left|\frac{0 - 1/5}{1 - (1/5) \cdot 0}\right| = 1/5.$$

For initial data z_1, z_2, and target data w_1, w_2 we know from the Schwarz-Pick theorem that if an $f \in \operatorname{End}(\mathbb{D})$ satisfies the interpolation problem, we must have

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This occurs if and only if the matrix,

$$\begin{pmatrix} \frac{1-|w_1|^2}{1-|z_1|^2} & \frac{1-w_1\overline{w}_2}{1-z_1\overline{z}_2} \\ \frac{1-w_2\overline{w}_1}{1-z_2\overline{z}_1} & \frac{1-|w_2|^2}{1-\overline{z_2}^2} \end{pmatrix}$$

is positive semidefinite. That is, if and only if it has nonnegative determinant.

Theorem

(Nevanlinna-Pick) Given initial data $z_1, z_2, \ldots, z_n$ *and target data* $w_1, w_2, \ldots, w_n$*, there exists* $f \in \mathrm{End}\,(\mathbb{D})$ *satisfying the data if and only if the matrix,*

$$\left(\frac{1-w_j\overline{w}_k}{1-z_j\overline{z}_k}\right)_{j,k=1}^{n}$$

is positive semidefinite.

Note that Sylvester's criterion says this is equivalent to the condition that all the determinants,

$$D_m = \left|\frac{1-w_j\overline{w}_k}{1-z_j\overline{z}_k}\right|_{j,k=1}^{m}$$

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Has application in control theory (see Allen Tannenbaum wikipedia).