

Week 1 Measure Theory Exercises

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Exercise 1

Exercise: Let $\mathcal{A}$ be an algebra of subsets of X . Prove that $\mathcal{A}$ is a σ -algebra if and only if it is closed under increasing countable unions.

Solution:

Proof. By definition, if $\mathcal{A}$ is a σ -algebra, it is closed under countable unions, including increasing countable unions, so it remains to show the other direction. Let $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}$ be an arbitrary sequence of sets. Define

$$B_n = \bigcup_{m=1}^n A_m$$

Clearly $B_n \subseteq B_{n+1}$, so $\{B_n\}_{n \in \mathbb{N}}$ is a countable increasing sequence, and since $\mathcal{A}$ is an algebra and each B_n is formed by a finite union of members of $\mathcal{A}$, each B_n is in $\mathcal{A}$. We have

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \in \mathcal{A}.$$

□

Exercise 2

Exercise: Let X be a non-empty set and let m^* be an outer measure on X with $m^*(X) < \infty$. Let $\nu^*(E) = (m^*(E))^{1/2}$ for all $E \subseteq X$. Prove that ν^* is also an outer measure. Prove that a set E belongs to the Carathéodory σ -algebra of ν^* iff $\nu^*(E)\nu^*(X \setminus E) = 0$.

Solution: First, we show that ν^* is also an outer measure.

Proof. First, we see that $\nu^*(\emptyset) = (m^*(\emptyset))^{1/2} = 0^{1/2} = 0$. Next, for $A \subseteq B$, we know that $\mu^*(A) \leq \mu^*(B)$ by the properties of an outer measure. Since the square-root function is monotone increasing, this means $(\mu^*(A))^{1/2} \leq (\mu^*(B))^{1/2}$, so $\nu^*(A) \leq \nu^*(B)$. Finally, we use the subadditivity of the square root function to see that

$$\nu^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \left(m^*\left(\bigcup_{n=1}^{\infty} A_n\right)\right)^{1/2} \leq \left(\sum_{n=1}^{\infty} m^*(A_n)\right)^{1/2} \leq \sum_{n=1}^{\infty} (m^*(A_n))^{1/2} = \sum_{n=1}^{\infty} \nu^*(A_n)$$

□

Now, we show that E belongs to the Carathéodory σ -algebra of ν^* , $\mathcal{M}(\nu^*)$, if and only if either $\nu^*(E) = 0$ or $\nu^*(X \setminus E) = 0$. Of course, if $\nu^*(E)$ or $\nu^*(X \setminus E)$ are zero, then they belong to this σ -algebra, so it remains to show the other direction.

Proof. Suppose $E \in \mathcal{M}(\nu^*)$. Then, we must have $\nu^*(X) = \nu^*(X \cap E) + \nu^*(X \setminus E)$. Squaring both sides, we see that $m^*(X) = m^*(X \cap E) + m^*(X \setminus E) + 2\nu^*(X \cap E)\nu^*(X \setminus E) = m^*(X \cap E) + m^*(X \setminus E)$, and since $X \cap E = E$, this means either $\nu^*(E) = 0$ or $\nu^*(X \setminus E) = 0$. □

Exercise 3

Exercise: Assume $E \subseteq \mathbb{R}$ has Lebesgue outer measure 0, i.e. $\lambda_1^*(E) = 0$, where λ_1^* is the one-dimensional Lebesgue outer measure. Prove that $\mathbb{R} \setminus E$ is dense in $\mathbb{R}$. Also prove that for all $A \subseteq \mathbb{R}$, we have $\lambda_1^*(A \cup E) = \lambda_1^*(A) = \lambda_1^*(A \setminus E)$.

Solution: First, we show that $\mathbb{R} \setminus E$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ with $x < y$ (so $\lambda_1^*((x, y)) > 0$). We must have $(x, y) \cap (\mathbb{R} \setminus E) \neq \emptyset$. To see why, suppose not; then $(x, y) \subseteq E$, and by monotonicity $\lambda_1^*(E) \geq \lambda_1^*((x, y)) > 0$, contradicting $\lambda_1^*(E) = 0$. Since x and y were arbitrary and $(x, y) \cap (\mathbb{R} \setminus E) \neq \emptyset$, $\mathbb{R} \setminus E$ is dense in $\mathbb{R}$. □

Now, we show that for all $A \subseteq \mathbb{R}$ that $\lambda_1^*(A \cup E) = \lambda_1^*(A) = \lambda_1^*(A \setminus E)$.

Proof. First we show the equality $\lambda_1^*(A \cup E) = \lambda_1^*(A)$. By subadditivity, we have

$$\lambda_1^*(A \cup E) \leq \lambda_1^*(A) + \lambda_1^*(E) = \lambda_1^*(A)$$

and since $A \subseteq A \cup E$ we have by monotonicity that $\lambda_1^*(A) \leq \lambda_1^*(A \cup E)$, showing that $\lambda_1^*(A \cup E) = \lambda_1^*(A)$. To show the equality $\lambda_1^*(A) = \lambda_1^*(A \setminus E)$, we first note that since $A \setminus E \subseteq A$, $\lambda_1^*(A \setminus E) \leq \lambda_1^*(A)$ by monotonicity. We can also see that $(A \setminus E) \cup (E \cap A) = A$, so by subadditivity

$$\lambda_1^*(A) = \lambda_1^*((A \setminus E) \cup (E \cap A)) \leq \lambda_1^*(A \setminus E) + \lambda_1^*(E \cap A) = \lambda_1^*(A \setminus E)$$

(we know $\lambda_1^*(E \cap A) = 0$ because $E \cap A \subseteq E$). Therefore, $\lambda_1^*(A \setminus E) = \lambda_1^*(A)$. □

Exercise 4

Exercise: For all $\delta \in (0, 1)$ we construct a set C_δ in a similar manner to the Cantor set. We start from $[0, 1]$ and we remove from every remaining interval of the k -th step a middle interval of length $\delta 3^{-k}$. Prove that C_δ is compact and with no isolated points. Prove that it does not contain an interval and prove that $\lambda_1(C_\delta) = 1 - \delta > 0$.

Solution: Compactness is easy to show: we take the closed interval $[0, 1]$ and take away countably many open intervals (which is the same as intersecting with the complement of the interval, which is closed). As C_δ is a countable intersection of closed sets, and it is bounded, it must be compact. To show that $\lambda_1(C_\delta) = 0$, we consider that at the k -step, there are 2^{k-1} intervals remaining, and so we obtain

$$\begin{aligned} \lambda_1(C_\delta) &= 1 - \sum_{k=1}^{\infty} 2^{k-1} (\delta 3^{-k}) = 1 - \frac{\delta}{2} \sum_{k=1}^{\infty} 2^k 3^{-k} = 1 - \frac{\delta}{2} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k = 1 - \frac{\delta}{2} \left(\frac{2/3}{1/3}\right) = 1 - \frac{\delta}{2}(2) \\ &= 1 - \delta. \end{aligned}$$

Now, we show that C_δ does not contain an interval. After the k -th step, notice that each of the remaining 2^k intervals have an equal length, which is

$$\frac{1}{2^k} \left(1 - \frac{\delta}{2} \sum_{k'=1}^k \left(\frac{2}{3}\right)^{k'} \right) = \frac{1 - \delta}{2^k} + \frac{\delta}{3^k} \xrightarrow{k \rightarrow \infty} 0$$

Since the length of all remaining intervals tends to zero as $k \rightarrow \infty$, C_δ cannot contain any intervals. A similar argument shows that C_δ does not contain any isolated points: let $x \in C_\delta$ and $\epsilon > 0$. Then, since the length of each interval remaining after step k tends towards zero as $k \rightarrow \infty$, there exists a step where the length of each remaining interval has length $< \epsilon$. x must be contained in some interval, and is at most ϵ away from the endpoints of this interval (which are also in C_δ), so x is not an isolated point of C_δ .