Octonionic Determinants

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The Ostensible Impasse

Recall the following:

- The familiar notion of determinant rests upon the commutativity and associativity of the underlying field.
- The Octonions are an 8-dimensional, non-associative, non-commutative algebra over the reals.

Bridging the Chasm… Halfway

- For square matrices over an associative (but not necessarily commutative) ring, we have at our disposal Gelfand and Retakh's quasideterminant.
- Unlike the determinant, which assigns a value to the entire matrix, *A*, one quasideterminant (when it exists) is assigned to each (i,j) entry and is denoted $[A]_{i,j}$
- In the commutative case where det(A) exists, we have

$$[A]_{i,j} = (-1)^{i+j}\frac{det(A)}{det(A^{i,j})}$$

Quasideterminant Definition

- Let A be an $n \times n$ matrix over a ring, $\boldsymbol{R}$ (not necessarily commutative).
- Suppose that A may be row reduced to the form

$$\begin{pmatrix} I_{n-1} & b \\ 0 & c \end{pmatrix}$$

such that
  - No rows are interchanged.
  - The nth is not scaled.
  - The nth row is not added to any other.
- Then $[A]_{n,n} = c$.
- If P permutes the i^{th} and nth rows and Q the j^{th} and n^{th} columns then $[A]_{i,j} = [PAQ]_{n,n}$ (provided that the latter exists)

Implementing the Quasideterminant

- Before applying quasideterminants to Octonionic matrices, we must first somehow address the non-associativity of Octonion multiplication.
- We may circumvent this problem altogether by restricting our attention to cases in which non-associativity is a moot point.

Hermitian Matrices Over O

- Let *A* be a 2 by 2 Hermitian matrix. Then

$$A = \begin{pmatrix} a & x \\ \overline{x} & b \end{pmatrix} \; with \; a, b \in \mathbb{R}, x \in \mathbb{O}$$

- We will compute $[A]_{2,2}$ directly:

$$\begin{pmatrix} a & x \\ \overline{x} & b \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & a^{-1}x \\ \overline{x} & b \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & a^{-1}x \\ 0 & b - \overline{x}(a^{-1}x) \end{pmatrix}$$

$$[A]_{2,2} = b - \overline{x}(a^{-1}x)$$

Exploiting Algebraic Properties of O

- We have

$$[A]_{2,2} = b - \overline{x}(a^{-1}x)$$

- Since **O** is a real algebra and simultaneously contains **R** as a subalgebra, any real number commutes and associates with all of O. Hence

$$\begin{aligned}[A]_{2,2} &= b - (\overline{x}a^{-1})x \\ &= b - a^{-1}(\overline{x}x)\end{aligned}$$

- In view of the conjugate norm relationship, we have

$$[A]_{2,2} = b - a^{-1}\|x\|^2$$

From Quasi to "Total" Determinant

$$A = \begin{pmatrix} a & x \\ \overline{x} & b \end{pmatrix} \; with \; a, b \in \mathbb{R}, x \in \mathbb{O}$$

$$[A^{2,2}]_{1,1} = a \qquad [A]_{2,2} = b - a^{-1}\|x\|^2$$

- Recall that if **O** ***were*** commutative and associative, the following identity would apply:

$$[A]_{2,2} = \frac{det(A)}{det(A^{2,2})} = \frac{det(A)}{[A^{2,2}]_{1,1}}$$

- Upon multiplying through, we find that

$$\begin{aligned}[A^{2,2}]_{1,1}[A]_{2,2} &= a(b - a^{-1}\|x\|^2) \\ &= ab - \|x\|^2\end{aligned}$$

- We shall regard this function as **n**(A). To what extent does it resemble a commutative determinant?

- **Theorem:** Let A be a 2×2 Hermitian, Octonionic matrix. Then A has a unique inverse (under standard matrix multiplication) if and only if $\mathbf{n}(A) \neq 0$. Furthermore, A^{-1}, when it exists, is, itself, Hermitian.

3×3 Hermitian Matrices

- Under the multiplication rule, $A \cdot B = \frac{1}{2}(AB+BA)$, the space of 3x3 Hermitian matrices over the Octonions becomes a Jordan algebra.
- Jacobson showed that when *A* is of the form

$$A = \begin{pmatrix} \alpha_1 & a_3 & \overline{a_2} \\ \overline{a_3} & \alpha_2 & a_1 \\ a_2 & \overline{a_1} & \alpha_3 \end{pmatrix} \; where \; \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}, a_1, a_2, a_3 \in \mathbb{O}$$

A has minimal polynomial whose constant term is given by

$$\mathbf{n}(A) = \alpha_1\alpha_2\alpha_3 + 2\mathbf{Re}((a_3a_1)a_2) - \sum_{i=0}^{3} \alpha_i \|a_i\|^2$$

n(*A*) Recast

- As **n** is the best candidate for "determinant," we ask whether we may express it in terms of quasideterminants.

$$
\begin{aligned}
[A^{32,32}]_{1,1} &= \alpha_1 \\
[A^{3,3}]_{2,2} &= \alpha_2 - \alpha_1^{-1}\|a_3\|^2 \\
[A]_{3,3} &= (\alpha_3 - \alpha_1^{-1}\|a_2\|^2) - (\alpha_2 - \alpha_1^{-1}\|a_3\|^2)^{-1}\|\overline{a_1} - \alpha_1^{-1}a_2a_3\|^2
\end{aligned}
$$

- Indeed, it is the case that

$$\mathbf{n}(A) = [A^{32,32}]_{1,1}[A^{3,3}]_{2,2}[A]_{3,3}$$