By Lemma 4, the coefficient of $\prod_{x=1}^{n} x_i^{d_i}$ in f_G is nonzero, because we assume that $EE(D) \neq EO(D)$. Finally we should prove, that the coefficient of this monomial cannot be affected by substitutions we have performed.

The degree of every monomial of f_G is equal in the number of edges of G. If we substitute (2.1) for $x_i^{d_i+1}$ to any monomial of f_G we obtain d_i monomials with strictly smaller degree. But such new monomials cannot affect coefficient of $\prod_{x=1}^{n} x_i^{d_i}$ because they have different degree. Thus the coefficient of $\prod_{x=1}^{n} x_i^{d_i}$ in f'_G is the same as in f_G. In particular, it is non-zero. Since we have obtained contradiction there exists a proper coloring $c : V \to \mathbb{Z}$ such that$c(v_i) \in S_i$.

3 The list chormatic number of the square of cycles

In this chapter, we will prove that the list chromatic number of the square of a cycle is equal to its chromatic number. Let us start with some definitions. Recall that $C_{\ell_0}^2$ denotes the square of the cycle of length ℓ_0.

We state an easy observation on Eulerian digraphs.

Proposition 3. *Let G be and Eulerian digraph and H one of its subgraph. If H is Eulerian, then $G - H$ i.e. the digraph obtained by removing the edges of H, is also Eulerian .*

Proof. Since G is Eulerian, the in-degree and out-degree of each vertex of G is the same. This also holds for H. Hence, there is the same number of ingoing and outgoing arcs removed at each vertex of G and thus the in-degree and the out-degree of each vertex are the same in $G - H$. Therefore, $G - H$ is Eulerian.

□

A direct corollary of Proposition 3 is the following.

Proposition 4. *Every subgraph of $C_{\ell_0}^2$ obtained by substracting an Eulerian subgraph from $C_{\ell_0}^2$ is Eulerian.*

Proposition 3 yields a correspondence between Eulerian subgraphs of $C_{\ell_0}^2$ with few and with a lot of edges. Let $\mathcal{S}$ denotes the set of Eulerian subgraphs of $C_{\ell_0}^2$ with less than ℓ_0 edges, $\mathcal{M}$ are subgraphs with exactly ℓ_0 edges and finally $\mathcal{L}$ is the set of subgraphs with more than ℓ_0 edges. It is easy to prove

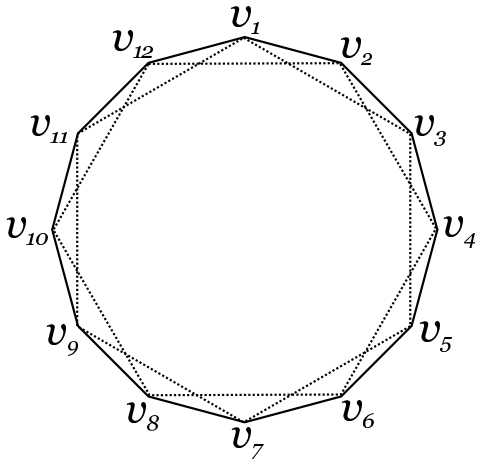

Figure 5: solid : jump by one, dashed : jump by two

that the mapping $\varphi : \mathcal{L} \longrightarrow \mathcal{S}$ given by $\varphi(G) = C^2_{\ell_0} - G$ is bijection. Since $C^2_{\ell_0}$ has an even number of edges, the bijection φ brings an even subgraph to an even subgraph and odd subgraph to odd subgraph. In particulary, the numbers of odd subgraphs in $\mathcal{S}$ and in $\mathcal{L}$ are equal. The same holds for even subgraphs.

Proposition 5. *Every non-empty Eulerian digraph contains at least one cycle.*

Proof. Consider a longest oriented path in G. Let $v_1, v_2, \ldots, v_k$ be this path (clearly $k > 2$). Since G is Eulerian, v_k has another neighbour, say w. By the choice of the path, w is one of the vertices v_i $(i \in 1, 2, \ldots, k-1)$. Then $v_i, v_{i+1}, \ldots, v_k$ is a cycle. □

Proposition 6. *Every subgraph H of $C^2_{\ell_0}$ contained in $\mathcal{S}$ is a single oriented cycle.*

Proof. If the vertices of $C^2_{\ell_0}$ are numbered from 1 to ℓ_0 then the arcs of $C^2_{\ell_0}$ "jump" by one or two (figure 5). Hence the length of the shortest cycle in $C^2_{\ell_0}$ is $\lceil \ell_0/2 \rceil$. Assume the opposite, that H contains two cycles. If we remove one of these cycles (which is also Eulerian subgraph of $C^2_{\ell_0}$), we obtain again Eulerian subgraph of $C^2_{\ell_0}$ (Proposition 4). But such graph would have less than $\lceil \ell_0/2 \rceil$ edges and it is impossible while every Eulerian graph contains at least one cycle and the shortest length of cycle is $\lceil \ell_0/2 \rceil$. □

Claim 1. *There are exactly two Eulerian subgraphs H of $C^2_{\ell_0}$ with ℓ_0 edges. In particular $|\mathcal{M}| = 2$.*

Proof. Using the notation from figure 5 one such graph is obviously cycle $v_1, v_2, v_3, \ldots, v_{\ell_0}$. The other subgraph is the cycle $v_1, v_3, \ldots, v_{\ell_0}, v_2, \ldots, v_{\ell_0-1}$ if ℓ_0 is odd and the union of two cycles $v_1, v_3, \ldots, v_{\ell_0-1}$ and $v_2, v_4, \ldots, v_{\ell_0}$ if ℓ_0 is even. In order to show that there are no other subgraphs we distinguish two cases based on the parity of ℓ_0. If ℓ_0 is odd there is exactly one cycle. We prove this analogously to Proposition 6. Assume that H contains two cycles. While the smallest cycle's length is $\lceil \ell_0/2 \rceil$, if we remove this cycle forms H, we obtain a graph with less than $\lceil \ell_0/2 \rceil$ arcs which is contradiction. This implies that any vertex has no more than one incoming and one outgoing arc and because H has ℓ_0 arcs it contains all vertices. We now prove that every subgraph H of $C^2_{\ell_0}$ constains either all arcs of C_{ℓ_0} or no arcs of C_{ℓ_0}. Otherwise H contains a path v_i, v_{i+1}, v_{i+3} or a path v_i, v_{i+2}, v_{i+3}. In the former case, $\deg^+_{v_{i+2}}$ has to be equal to 0 which contradicts that all vertices have non-zero degrees in H. In the latter case, there cannot be any arc leaving from v_{i+1} and we again obtain a contradiction.

Assume now that ℓ_0 is even. If H is connected, i.e., H is a single cycle, we argue as in the case where ℓ_0 is odd that H is the cycle $v_1, v_2, \ldots, v_{\ell_0}$. Otherwise, H is comprised of two cycles of length $\ell_0/2$ each. Such cycles must be formed by "jump-by-two"-edges. If they had a common vertex, they would be identical. Hence they are disjoint and consequently H must be the union of the cycles $v_1, v_3, \ldots$ and $v_2, v_4, \ldots$. □

By the bijection φ between $\mathcal{S}$ and $\mathcal{L}$ it is enough to analyze the number of of even and odd subgraphs of $C^2_{\ell_0}$ in $\mathcal{S}$. By Proposition 6, any subgraph contained in $\mathcal{S}$ is an oriented cycle. Such a cycle can be identified with a cyclic sequence of length ℓ_0 of zeroes and ones with no two consequentive zeroes as follows: the $i-th$ position is equal to 1 if the vertex v_i is not an isolated vertex of the considered subgraph of $C^2_{\ell_0}$.

This allows us to reformulate the problem to counting sequences with odd and even number of ones.

In the final part of proof, we show how to count the number of such sequences. Defina odd sequences as the sequences with odd number ones and even sequences are the sequences with even nubmer of ones. We also find a formula for transforming number of sequences to number of graphs.

We start with a few definitions.

Definition 2. *Let o_n^1 denote the number of sequences of length n containing an odd number of ones that end with one. Similarly, let o_n^0 be the number of sequences with an odd number of ones that end with* 0*. Analogously, we define e_n^1 and e_n^0 for sequences with even number of ones.*

It is easy to infer the following recurent formulas:

$$\begin{aligned} o_n^1 &= e_{n-1}^1 + e_{n-1}^0 \\ o_n^0 &= o_{n-1}^1 \\ e_n^1 &= o_{n-1}^1 + o_{n-1}^0 \\ e_n^0 &= e_{n-1}^1 \end{aligned}$$

The following equation exhibit the correspondence between the number of the sequences and the number N_e of Eulerian subgraphs of G with even and odd number of arcs. The number of even Eulerian subgraphs of G where $|V(G)| = n$ is

$$N_e = e_n^1 + e_n^0 - o_{n-2}^0 \tag{3.1}$$

Analogously for the odd subgraphs of G

$$N_o = o_n^1 + o_n^0 - e_{n-2}^0 \tag{3.2}$$

First, we prove the equation 3.2. It is not possible sum o_n^1 and o_n^0, because these sequences are not cyclic and we can obtain illegal subgraphs of G (every cyclic sequence corresponds to exactly one Eulerian subgraph of G), since it could contain two consecutive zeros. Note that this problem is caused by sequences starting with 01 and ending with 0. But such sequences of length n can be written as $01\sigma0$, where σ stands for an even sequence of length $n-2$ ending with zero. Hence, the number of such sequences is e_{n-2}^0. Analogously, we derive that the number of even subgraphs of G is equal to $e_n^1+e_n^0-o_{n-2}^0+1$, where the added one stands for empty graph.

In the following lemma we show that the numbers of odd and even Eulerian subgraphs of G differ and so we can use Alon-Tarsi theorem.

Lemma 5. *The number of odd sequences of length n and even sequences of length n differs for $n = 3 * k$.*

Proof. First we prove that the following equations hold:

$$o_n^1 = e_n^1 - 1$$

$$o_n^0 = e_n^0$$

for n divisible by three.

Let us have a brief look at th values of o_n^1, o_n^0, e_n^1 and e_n^0 for $n = 1, \ldots, 6$.

$$o_n^1 : 1, 1, 1, 3, 4, 6$$
$$o_n^0 : 0, 1, 1, 1, 3, 4$$
$$e_n^1 : 0, 1, 2, 2, 4, 7$$
$$e_n^0 : 1, 0, 1, 2, 2, 4$$

If n=3 or n=6, the claimed equations are true. Assume the equations in question hold for n let us prove them for n+3. We infer the following from the induction:

$$o_{n+1}^1 = e_n^1 + e_n^0 = o_n^1 + 1 + o_n^0 \wedge e_{n+1}^1 = o_n^1 + o_n^0 \Rightarrow o_{n+1}^1 = e_{n+1}^1 + 1$$

By applying the recurence formula once more, we obtain

$$o_{n+1}^0 = e_{n+1}^0 - 1$$

and by yet another application of the recurence formula, we get

$$o_{n+2}^1 = e_{n+2}^1$$
$$o_{n+2}^0 = e_{n+2}^0 + 1$$

and then

$$o_{n+3}^1 = e_{n+3}^1 - 1$$
$$o_{n+3}^0 = e_{n+3}^0$$

This finishes the proof of the equations.

Recall, that the number of odd Eulerian subgraphs of G is equal to

$$o_n^1 + o_n^1 - e_{n-2}^0.$$

After substituting for o_n^1 and o_n^1 using equations from previous paragraph we obtain

$$o_n^1 + o_n^0 - e_{n-2}^0 = e_n^1 - 1 + e_n^0 - (o_{n-2}^0 + 1) = e_n^1 + e_n^0 - o_{n-2}^0 + 1 - 3.$$

This can readily be interpreted as

$$N_o = N_e - 3$$

which finishes the proof.

□

The result is that we have found one such orientation D of G, that number of even Eulerian subgraphs is not equal to the number of odd Eulerian subgraph, which is enough to state, that there exists proper coloring. Moreover, because the largest out-degree is 2, we know, that the size of lists is 3.

□

4 Application of Alon-Tarsi Theorem

In this chapter, we present as an application of Alon-Tarsi Theorem the following result of Fleischner and Stiebitz [4] which answers a problem posed by Erdős at the Julius Petersen Graph Theory Conference in 1990.

Theorem 3. *Let n be a positive integer, and let G be a 4-regular graph on $3n$ vertices. Assume that G has a decomposition into a Hamilton cycle and n pairwise vertex disjoint triangles. Then $\chi(G) = 3$.*

Before proving this theorem, we introduce some notation which was not explained yet. Let D be a digraph. If E is a subset of the edge-set $E(D)$ such that the digraph $(V(D), E)$ is Eulerian, then E is called an *Eulerian set of arcs*. Moreover, we use the following notation

$$\varepsilon(D) := \{E \subseteq E(D), E \text{ is an Eulerian set of arcs in } D\},$$

$$\varepsilon_e(D) := \{E \subseteq E(D), |E| \text{ is even}\}, \text{ and}$$
$$\varepsilon_o(D) := \{E \subseteq E(D), |E| \text{ is odd}\}.$$

The cardinalities of these three sets are denoted by $e(D)$, $e_e(D)$ and $e_o(D)$, respectively.

The set of the outgoing arcs from a vertex v in the graph G is denoted by $E^+_G(v)$ and the set of the incoming arcs by $E^-_G(v)$. Finally $E_G(v)$ is the union of the sets $E^+_G(v)$ and $E^-_G(v)$.

We now present notation related to Eulerian subdigraphs of Eulerian digraphs. If D is a digraph, $a_1, \ldots, a_p, b_1, \ldots, b_q$ are some of its arcs and $x_1, \ldots, x_r, y_1, \ldots, y_s$ are some of its vertices, then

$$\varepsilon := \varepsilon(D, a_1, \ldots, a_p, \overline{b_1}, \ldots, \overline{b_q}, x_1, \ldots x_r, \overline{y_1}, \ldots, \overline{y_s})$$

denotes the set of Eulerian sets of arcs E such that $a_1, \ldots, a_p \in E, b_1, \ldots, b_q \notin E$, $E_G(x_i) \subseteq E$, $i = 1, \ldots, r$ and $E_G(y_i) \cap E = \emptyset$, $i = 1, \ldots, s$. Furthermore, set

$$e(D, a_1, \ldots, a_p, \overline{b_1}, \ldots, \overline{b_q}, x_1, \ldots x_r, \overline{y_1}, \ldots, \overline{y_s}) := |\varepsilon|.$$

Finally, the set E^R is defined as the set of the arcs xy such that $yx \in E$, i.e. E^R is the set of the arcs E with their orientations reversed.

As the next step towards the proof of the result of Fleischner and Stiebitz, we establish three lemmas.

Lemma 6. *Let D be an Eulerian digraph with $m \geq 1$ arcs. If φ is a mapping defined as $\varphi(E) := E(D) \setminus E$, then the following statements hold:*

1. *The mapping φ is a bijection from $\varepsilon(D)$ onto itself without a fixed point. In particular, $e(D) \equiv 0 \ mod\ 2$.*

2. *For $a_1, \ldots, a_p, b_1, \ldots, b_q \in E(D)$ and $x_1, \ldots, x_r, y_1, \ldots, y_s \in V(D)$, the mapping φ is also a bijection from*

$$\varepsilon(D, a_1, \ldots, a_p, \overline{b_1}, \ldots, \overline{b_q}, x_1, \ldots x_r, \overline{y_1}, \ldots, \overline{y_s}) \text{ onto}$$

$$\varepsilon(D, \overline{a_1}, \ldots, \overline{a_p}, b_1, \ldots, b_q, \overline{x_1}, \ldots \overline{x_r}, y_1, \ldots, y_s).$$

   Consequently,

$$e(D, a_1, \ldots, a_p, \overline{b_1}, \ldots, \overline{b_q}, x_1, \ldots x_r, \overline{y_1}, \ldots, \overline{y_s})$$

$$= e(D, \overline{a_1}, \ldots, \overline{a_p}, b_1, \ldots, b_q, \overline{x_1}, \ldots \overline{x_r}, y_1, \ldots, y_s).$$

3. *If m is odd, then φ is a bijection from $\varepsilon_o(D)$ onto $\varepsilon_e(D)$, and $\varepsilon_o(D) = \varepsilon_e(D)$.*

4. *If m is even, then φ is bijection from $\varepsilon_o(D)$ onto itself and $\varepsilon_e(D)$ onto itself, and $\varepsilon_o(D) \equiv \varepsilon_e(D) \equiv 0 \ mod\ 2$.*

5. *If m is even and $e(D) \equiv 2 \ mod\ 4$, then $\varepsilon_o(D) \neq \varepsilon_e(D)$.*

Proof. 1. The mapping φ is a bijection from $\varepsilon(D)$ to $\varepsilon(D)$ by Proposition 4. Clearly, the bijection has no fixed point. Since $\varphi^{-1} = \varphi$, φ couples Eulerian sets of arcs of D. Hence $e(D)$ is even.

2. The claim follows from the definition of the set $\varepsilon(D, a_1, \ldots, \overline{y_s})$

3. If m is odd, then φ maps odd-size Eulerian sets of arcs to even-size ones by its definition.

4. As in the previous case, φ is a bijection from $\varepsilon_e(D)$ onto itself. Since φ has no fixed point and $\varphi^{-1} = \varphi$, $e_e(D)$ is even. Similarly we can argue in the odd case.

5. Since $e(D) \equiv 2 \bmod 4$, $e_o(D)$ and $e_e(D)$ are both even by the previous claim, the numbers $e_e(D)$ and $e_o(D)$ cannot be congruent modulo 4. In particular, they are different.

□

Lemma 7. *Let D be an Eulerian digraph and C a (directed) cycle of length m in D. Set D_1 to be a digraph $(D-C)\cup(C)^R$. Let the mapping φ_C is defined as $\varphi_C(E) := (E \setminus E(C)) \cup (E(C) \setminus E)^R$ for an eulerian arc set $E \in \varepsilon(D)$. Then the following statements hold:*

1. *D_1 is an Eulerian digraph.*

2. *The mapping φ_C is a bijection from $\varepsilon(D)$ onto $\varepsilon(D_1)$. In particular, $e(D) = e(D_1)$.*

3. *If m is odd, then φ_C is a bijection from $\varepsilon_o(D)$ onto $\varepsilon_e(D_1)$. Symetrically, it is also a bijection from $\varepsilon_e(D)$ onto $\varepsilon_o(D)$. Consequently, $e_o(D) - e_e(D) = e_e(D_1) - e_o(D_1)$.*

4. *If m is even, then φ_C is a bijection from $\varepsilon_o(D)$ onto $\varepsilon_o(D_1)$ and it is also a bijection from $\varepsilon_e(D)$ onto $\varepsilon_e(D_1)$. Hence, $e_o(D) - e_e(D) = e_o(D_1) - e_e(D_1)$.*

Proof. 1. We observe that every vertex v in D_1 has the same in-degree and out-degree as in D. If v is not contained in C, then the in-degree and the out-degree are clearly preserved. If v is contained in C, then one arc incoming to v is changed to an outgoing arc and one arc outgoing from v is changed to an incoming arc. Hence, the in-degree and the out-degree of v are unchanged, and thus D_1 is Eulerian.

2. Let us fix a set $E \in \varepsilon(D)$. D' denotes the Eulerian subgraph of D consisting of the edges E and D'_1 denotes the Eulerian subgraph of D with the edge set $\varphi_C(E)$. We first prove that D'_1 is also an Eulerian digraph. We distinguish two cases based on whether a vertex x of D

is in C or not. The latter case is easier to analyze, since it is enough to realize that the set $E_{D'}(x)$ is unchanged. Because D' is an Eulerian digraph, it holds that $deg^+_{D'_1}(x) = deg^-_{D'_1}(x)$. If x belongs to C, then we further distinguish three subcases. If $E_{D'}(x) \cap E(C) = \emptyset$, then $E_{D'_1}(x)$ contains in addition the two arcs of C^R incident with x. Hence, $deg^+_{D'_1}(x) = deg^-_{D'_1}(x)$. If $|E_{D'}(x) \cap E(C)| = 2$, then the two arcs of C are removed from $E_{D'}(x)$ to get $E_{D'_1}$ and thus $deg^+_{D'_1}(x) = deg^-_{D'_1}(x)$.

Finally assume that $|E_{D'}(x) \cap E(C)| = 1$. Let $e_1, e_2 \in E(C)$ be the edges incident with x. We can assume that $e_1 \in D'$ and $e_2 \notin D'$ by symetry. By the definition of φ_C, e_1 is removed from D' and e_2^R is added to D'_1. Since one of these edges is incoming to x in D' and the another one is outgoing from x in D', the mapping φ_C does not change the sizes of the sets $E^+_{D'}(x)$ and $E^-_{D'}(x)$. This implies that $deg^+_{D'_1}(x) = deg^-_{D'_1}(x)$.

Observe that the mapping φ_C from $\varepsilon(D_1)$ is the inverse mapping for φ_C. Hence, φ_C is a bijection between $\varepsilon(D)$ and $\varepsilon(D_1)$.

3. Assume that the cardinality of the intersection $E \cap E(C)$ is odd, then the cardinality of $E(C) \setminus E$ is even since the length of C is odd. In particular, the parity of the size of the set $E \setminus E(C)$ is different from the parity of $|E|$. Hence, $|E| \not\equiv |\varphi_C(E)| \bmod 2$. We obtain that $\varphi_C(E)$ is a bijection from $\varepsilon_e(D)$ onto $\varepsilon_o(D_1)$ and also from $\varepsilon_o(D)$ onto $\varepsilon_e(D_1)$. This implies that $e_o(D) - e_e(D) = e_e(D_1) - e_o(D_1)$.

4. The proof of this claim proceed alongs the lines of the proof of the previous claim.

□

Recall that if D_1 and D_2 are two digraphs on the same vertex set, then $D_1 - D_2$ is the digraph containing the arcs of D_1 that are not present in D_2 with the same orientation.

Lemma 8. *Let D_1 and D_2 be Eulerian orientations of the Eulerian graph G. Set $D_0 := D_1 - D_2 = (D_2 - D_1)^R$, and $D_3 := D_1 \cap D_2$. D_0 and D_3 are Eulerian digraphs.*

Proof. We first prove that D_3 is an Eulerian digraph. Both orientations are Eulerian, thus $|E^+_{D_1}(v)| = |E^+_{D_2}(v)| = deg(v)/2$ for any vertex v of G. The number of outgoing arcs from v in D_1 with different orientation than in D_2

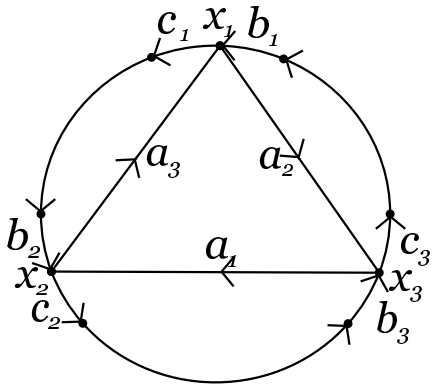

Figure 6: The graph D from the proof of Lemma 9.

is equal to the number of incoming arcs to v in D_1 with different orientation in D_2. Thus the number of incoming arcs to v in D_3 is equal to the number of outgoing arcs from v in D_3. It follows that D_3 is Eulerian.

Obviously $D_1 - D_2 = D_1 - D_3$. Lemma4 implies that D_0 is an Eulerian digraph. Observe that if D_1 and D_2 are orientations of the same graph, then $D_1 - D_2 = (D_2 - D_1)^R$. □

Now we are ready to prove the following lemma, which yields Theorem 3 as argued further.

Lemma 9. *Let D be an Eulerian digraph. Assume that D has a decomposition into a (directed) Hamilton cycle and $n \geq 0$ pairwise vertex disjoint (directed) triangles. Then $e(D) \equiv 2 \mod 4$.*

Proof of Theorem 3 using Lemma 9. Lemma 6.5 implies that $e_o(D) \neq e_e(D)$. Since D is Eulerian and every two triangles are pairwise disjoint, the out-degree of every vertex is at most 2 and thus we infer from Alon-Tarsi Theorem $\chi_l \leq 3$. □

Proof of Lemma 9. We prove this lemma by the induction on n. For $n = 0$ it holds $e(D) = 2$ since there are exactly two Eulerian subgraphs of D, an empty graph and D itself.

Let $n \geq 1$ and T be one of the triangles with vertices x_1, x_2 and x_3 and arcs a_1, a_2 and a_3 (see Figure 6).

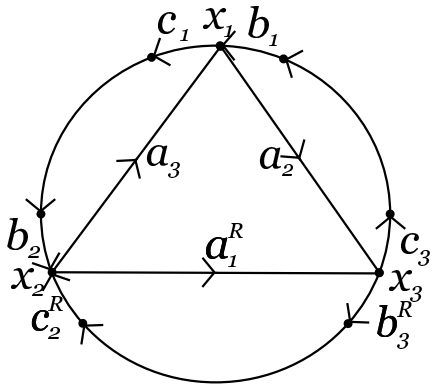

Figure 7: The graph D_i from the proof of Lemma 9.

Set

$$e^*(D) := e(D, \overline{a_1}, a_2, a_3) + e(D, a_1, \overline{a_2}, \overline{a_3}), + e(D, a_1, \overline{a_2}, a_3)$$

$$+ e(D, \overline{a_1}, a_2, \overline{a_3}) + e(D, a_1, a_2, \overline{a_3}) + e(D, \overline{a_1}, \overline{a_2}, a_3).$$

Consequently,

$$e(D) = e(D, a_1, a_2, a_3) + e(D, \overline{a_1}, \overline{a_2}, \overline{a_3}) + e^*(D).$$

Obviously, we can apply the induction on $e(D, \overline{a_1}, \overline{a_2}, \overline{a_3})$ since it contains $n - 1$ triangles and a Hamilton cycle. In addition, since T is a directed cycle, say $x_1x_3x_2$, if we add it to any Eulerian subgraph D' of the graph $D - T$, then we obtain an Eulerian graph and if we add T to any non-Eulerian graph, the resulting graph will also be non-Eulerian, which implies that $e(D, a_1, a_2, a_3) = e(D, \overline{a_1}, \overline{a_2}, \overline{a_3}) \equiv 2 \mod 4$. This implies

$$e(D) \equiv e^*(D) \mod 4.$$

Therefore it is enough to show that $e^*(D) \mod 4 \equiv 2 \mod 4$. We now construct three digraphs D'_1, D'_2 and D'_3 to which the induction can be used.

For $i = 1, 2, 3$, let x_i^+ be the succesor of x_i on C and x_i^- the predeccesor of x_i on C (see Figure 7). Finally, b_i denotes arc $x_i^-x_i$ and c_i denotes arc $x_ix_i^+$. By Lemma 2, reversing the orientation of C does not change $e(D)$. Hence, we assume that C is oriented from x_1 to x_2, from x_2 to x_3 and from x_3 to x_1 as in Figure 6. The cycle C_i denotes the cycle containing arc a_i and a part

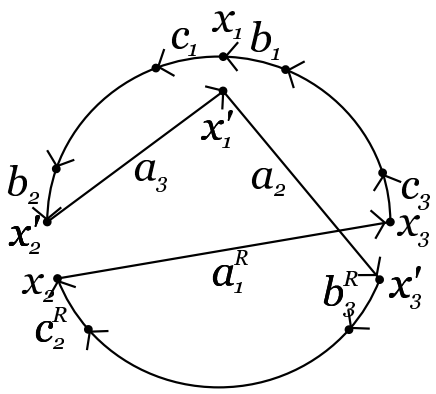

Figure 8: The graph D'_i from the proof of Lemma 9.

of C in such a way that all C_i are arc disjoint. If we reverse the orientation of the cycle C_i in D, we obtain the graph D_i (Figure 7).

In the following, (i, j, k) is one of the triples $(1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$ and we illustrate the proof for $(i, j, k) = (1, 2, 3)$ in the figures.

First we define a digraph D'_i to be a digraph obtained from D_i by splitting off the arcs b_j, a_k, a_j and b_k^R (Figure 8). The new vertices produced by splitting the arcs are x'_i, x'_j and x'_k as given in the figure. The obtained graph has $n - 1$ triangles and contains the Hamilton cycle. By the induction we get that

$$e(D'_1) \equiv e(D'_2) \equiv e(D'_3) \equiv 2 \mod 4, \tag{4.1}$$

which in turn implies

$$e(D'_1) + e(D'_2) + e(D'_3) \equiv 2 \mod 4.$$

Obviously, every Eulerian arc set $E \in \varepsilon(D'_i)$ contains either both arcs a_k or a_j or none of them, as the degree of x'_i is two. For $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$, it holds that

$$e(D'_i) = e(D'_i, a_i^R, a_j, a_k) + e(D'_i, \overline{a_i^R}, a_j, a_k) + e(D'_i, a_i^R, \overline{a_j}, \overline{a_k}) + e(D'_i, \overline{a_i^R}, \overline{a_j}, \overline{a_k}). \tag{4.2}$$

To find the relations between $e(D'_i)$ and $e(D)$ we start with proving the following equation

$$e(D'_i, a_i^R, a_j, a_k) = e(D, \overline{a_i}, a_j, a_k). \tag{4.3}$$

Set $\varepsilon' := \varepsilon(D'_i, a_i^R, a_j, a_k)$ and $\varepsilon = \varepsilon(D, \overline{a_i}, a_j, a_k)$. Now we prove that there exists a bijection between ε and ε' which yields the equation 4.3. First we observe that

$$\varepsilon' = \varepsilon(D'_i, a_i^R, a_j, a_k) = \varepsilon(D'_i, c_j^R, a_i^R, b_k^R, c_k, b_j, a_j, a_k).$$

It is easy to see that this equation holds since we just add necessary arcs to keep the in-degrees and out-degrees of x_j and x_k equal. By the same reason the following equation is true

$$\varepsilon = \varepsilon(D, \overline{a_i}, a_j, a_k) = \varepsilon(D, \overline{c_j}, \overline{a_i}, \overline{b_k}, c_k, b_j, a_j, a_k).$$

We next show that the following holds

$$\varepsilon' = \varepsilon(D'_i, c_j^R, a_i^R, b_k^R, c_k, b_j, a_j, a_k) = \varepsilon(D_i, c_j^R, a_i^R, b_k^R, c_k, b_j, a_j, a_k).$$

The question is whether every set of arcs which is Eulerian for D'_i is Eulerian for D_i. For splitted vertices it holds simply from definition of ε. For other vertices the in-degree and out-degree is obviously same in D_i and D'_i.

Let us have a mapping φ_{C_i} defined as $\varphi_{C_i}(E) := (E \setminus E(C_i)) \cup (E(C_i) \setminus E)^R$ for every $E \in \varepsilon(D)$. By Lemma 7 the mapping φ_{C_i} is a bijection from $\varepsilon(D)$ onto $\varepsilon(D_i)$. From the definition of φ_{C_i} Eulerian sets E in ε that avoid an arc $a \in E(C_i)$ are mapped to sets containing a^R and vice versa. The presence of arcs $a \notin E(C_i)$ in E is not affected by the mapping φ_{C_i}. Thus φ_{C_i} is mapping from ε to ε'. Again, mapping φ_{C_i} from ε' is the inverse mapping for φ_{C_i}. Thus φ_{C_i} is a bijection and the equation (4.3) holds.

It directly follows by Lemma 6.2 and (4.3) that

$$e(D'_i, \overline{a_i^R}, \overline{a_j}, \overline{a_k}) = e(D, a_i, \overline{a_j}, \overline{a_k}). \tag{4.4}$$

Before we state the next equation, let us assign $D' := D - T$ where T is the triangle $x_1x_2x_3$. Then it holds that

$$e(D'_i, a_i^R, \overline{a_j}, \overline{a_k}) = e(D', \overline{x_j}, x_k) = e(D', x_i, \overline{x_j}, x_k) + e(D', \overline{x_i}, \overline{x_j}, x_k). \tag{4.5}$$

Set $\varepsilon' := \varepsilon(D'_i, a_i^R, \overline{a_j}, \overline{a_k})$ and $\varepsilon := e(D', \overline{x_j}, x_k)$. Similarly as in (4.3) we obtain

$$\varepsilon' = \varepsilon(D'_i, c_j^R, a_i, c_k, \overline{b_j}, \overline{a_j}, \overline{a_k}, \overline{b_k^R}) = \varepsilon(D_i, c_j^R, a_i, c_k, \overline{b_j}, \overline{a_j}, \overline{a_k}, \overline{b_k^R}),$$

and consequently

$$\varepsilon = \varepsilon(D', \overline{b_j}, \overline{c_j}, b_k, c_k) = \varepsilon(D, \overline{a_j}, \overline{a_k}, \overline{b_j}, \overline{c_j}, b_k, c_k).$$

The last equation follows from the definition of D'. In the same way as in the proof of the equation (4.3), we get $e(D'_i, a_i^R, \overline{a_j}, \overline{a_k}) = e(D', \overline{x_j}, x_k)$, since there exists a bijection from ε onto ε'. Because the degree of x_i is two, the second equality in (4.5) holds an thus the equation (4.5) is proved.

We infer from (4.3) using Lemma 6.2 that

$$e(D'_i, \overline{a_i^R}, a_j, a_k) = e(D', x_j, \overline{x_k}) = e(D', x_i, x_j, \overline{x_k}) + e(D', \overline{x_i}, x_j, \overline{x_k}). \quad (4.6)$$

Set

$$m := e(D_1) + e(D_2) + e(D_3),$$

and

$$m' := \sum_{(i,j,k)\in\{(1,2,3),(2,3,1),(3,1,2)\}} e(D', x_i, \overline{x_j}, x_k) + e(D', \overline{x_i}, \overline{x_j}, x_k) + e(D', x_i, x_j, \overline{x_k})$$
$$+ e(D', \overline{x_i}, x_j, \overline{x_k}).$$

if we combine (4.2)-(4.6), we obtain $m = e^*(D) + m'$. Notice that each triple (i, j, k) in m' stands for one digraph D_i. Using Lemma 6.2 twice and rearranging the terms we conclude

$$m' := 2 * \sum_{(i,j,k)\in\{(1,2,3),(2,3,1),(3,1,2)\}} e(D', x_i, \overline{x_j}, x_k) + e(D', \overline{x_i}, \overline{x_j}, x_k)$$

and finally

$$m' := 4 * (e(D', x_1, x_2, \overline{x_3}) + e(D', x_1, \overline{x_2}, x_3) + e(D', \overline{x_1}, x_2, x_3)).$$

Hence, $m' \equiv 0 \mod 4$ which is equivalent to $m \equiv e^*(D) \mod 4$. We now infer from (4.1) that $e(D) \equiv 2 \mod 4$ which implies the assertion $e(D) \equiv 2 \mod 4$ of the lemma. □

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