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ON CERTAIN SUBGRAPHS OF A COMPLETE TRANSITIVELY DIRECTED GRAPH

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Let G_n be a complete transitively directed graph with $n + 1$ vertices $v_0, v_1, ..., v_n$. Let $\psi(n)$ be the number of subgraphs H of G_n where each vertex in H lies along a directed path from v_0 to v_n in H. $\psi(n)$ and some related quantities are calculated.

1. Introduction

In a paper on automata theory [1] the authors utilized directed simple graphs G_n, $n = 1, 2, ...$, defined as follows: G_n has vertices $v_0, v_1, ..., v_n$ and (v_i, v_j) is an arc if and only if $i < j$. (Thus G_n is a complete transitively directed graph with $n + 1$ vertices.)

The following problem arises in [1]: How many subgraphs of G_n possessing the upper and lower lattice property contain v_0 and v_n? Or equivalently, what is the number of nonempty connected subgraphs H of G_n such that each vertex in H lies along a directed path from v_0 to v_n in H?

The set of such subgraphs will be denoted Ω_n. We calculate card(Ω_n) (the cardinality of Ω_n), and some related quantities, establish some asymptotic expressions and calculate some generating polynomials.

2. Bounds and asymptotic expressions

It is convenient for our purposes to study also

$$\Omega'_n = \{H \in \Omega_n : H \text{ has vertices } v_0, v_1, ..., v_n\} .$$

Definition 2.1. $\mathrm{card}(\Omega_n) = \psi(n)$; $\mathrm{card}(\Omega'_n) = \xi(n)$.

One has immediately

$$(1) \qquad \psi(n) = \sum_{i=0}^{n-1} \binom{n-1}{i} \xi(n-i), \qquad \xi(n) = \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i \psi(n-i)$$

To establish bounds for $\xi(n)$ and $\psi(n)$ we notice that any graph in Ω_n is determined by its arcs. Since G_n has $\binom{n+1}{2}$ arcs, there are $2^{\binom{n+1}{2}}$ subsets of arcs from G_n. Any such subset containing $\{(v_i, v_{i+1}): i = 0, \ldots, n-1\}$ corresponds to a graph in Ω'_n. There are $2^{\binom{n+1}{2}-n} = 2^{\binom{n}{2}}$ such subsets.

Thus we have immediately

$$2^{\binom{n}{2}} \leqslant \xi(n) \leqslant \psi(n) \leqslant 2^{\binom{n+1}{2}} \quad \text{for all } n\,.$$

But this inequality can be improved. We can prove the following proposition. ([ ] denotes the greatest integer function.)

Proposition 2.2.

$$\prod_{i=1}^{n} (1 - 2^{-i})^2 \leqslant \frac{\xi(n)}{2^{\binom{n+1}{2}}} \leqslant \prod_{i=1}^{[n/2]} (1 - 2^{-i})^2\,.$$

Proof. We shall use probabilistic arguments. Let H be a set of arcs in G_n chosen at random, all choices having the same probability. Then the probability that H corresponds to a graph in Ω'_n is $\xi(n)/2^{\binom{n+1}{2}}$.

Let $0 < i \leqslant n$. The probability that v_i is a terminal endpoint of some arc in H is

$$p_i = 1 - 2^{-i}\,.$$

Hence the probability that all v_i, $i = 1, \ldots, n$, are terminal endpoints of arcs in H is

$$p = \prod_{i=1}^{n} (1 - 2^{-i})\,.$$

Similarly we find that the probability that each v_i, $i = 0, \ldots, n-1$, is an initial endpoint of some arc in H is

$$q = \prod_{i=1}^{n} (1 - 2^{-i}) .$$

Now if each point is a terminal endpoint, the probability that each point is an initial endpoint clearly *increases* from q to some number q'. Thus the probability r that H corresponds to a graph in Ω'_n satisfies

$$r \geqslant pq = \prod_{i=1}^{n} (1 - 2^{-i})^2 ,$$

which is the left inequality.

On the other hand, $r \leqslant r'$ = the probability that all points v_i, $1 \leqslant i \leqslant [\frac{1}{2}n]$, are terminal endpoints, all points v_j, $[\frac{1}{2}(n+1)] \leqslant j \leqslant n-1$, are initial endpoints of arcs in H. These two events are independent, and we get

$$r \leqslant r' = \prod_{i=1}^{[n/2]} (1 - 2^{-i})^2 ,$$

which is the right inequality.

To obtain bounds for $\psi(n)$ we use (1). We have for $n \geqslant 2$,

$$\frac{\psi(n)}{\xi(n)} = 1 + (n-1)\frac{\xi(n-1)}{\xi(n)} + \sum_{i=2}^{n-1} \binom{n-1}{i} \frac{\xi(n-i)}{\xi(n)} .$$

Noting that $\prod_{i=1}^{\infty}(1 - 2^{-i})^2 \simeq 0{,}0834 > 2^{-4}$ and that ξ (like ψ) is an increasing function, we have

$$\frac{\psi(n)}{\xi(n)} \leqslant 1 + (n-1)\frac{\xi(n-1)}{\xi(n)} + 2^{n-1}\frac{\xi(n-2)}{\xi(n)}$$

$$\leqslant 1 + (n-1)2^{-n} \cdot \prod_{i=[(n-1)/2]+1}^{n} (1 - 2^{-i})^{-2}$$

$$+ 2^{-n} \cdot \prod_{i=[(n-2)/2]+1}^{n} (1 - 2^{-i})^{-2}$$

$$\leqslant 1 + n \cdot 2^{-n} \cdot \prod_{i=1}^{\infty} (1 - 2^{-i})^{-2} < 1 + n \cdot 2^{-(n-4)} .$$

On the other hand,

$$\frac{\psi(n)}{\xi(n)} \geqslant 1 + (n-1)\frac{\xi(n-1)}{\xi(n)}$$

$$\geqslant 1 + (n-1)2^{-n} \prod_{i=[n/2]+1}^{n-1} (1-2^{-i})^2 \geqslant 1 + (n-1)2^{-n-4}.$$

Combining these results with Proposition 2.2, we obtain the following proposition.

Proposition 2.3. *For $n = 1, 2, \ldots$, we have*

$$(1 + (n-1)2^{-(n+4)}) \prod_{i=1}^{n} (1-2^{-i})^2 \leqslant$$

$$\leqslant \frac{\psi(n)}{2^{\binom{n+1}{2}}} \leqslant (1 + n \cdot 2^{-(n-4)}) \prod_{i=1}^{[n/2]} (1-2^{-i})^2 .$$

(The case $n = 1$ must be verified separately.)

Remark. It will follow from Proposition 4.1 that the upper bound in Proposition 2.3 can be replaced by

$$\prod_{i=1}^{[(n-1)/2]} (1-2^{-i})^2 .$$

Corollary 2.4. *$\psi(n)$ and $\xi(n)$ are both asymptotically equal to $C^2 2^{\binom{n+1}{2}}$, where*

$$C^2 = \prod_{i=1}^{\infty} (1-2^{-i})^2 \simeq 0{,}0834 .$$

3. Exact formulas

The adjacency matrix of G_n is $((c_{ij}))_{i,j=0,\ldots,n}$, where $c_{ij} = 0$ if $i \geqslant j$, $c_{ij} = 1$ otherwise. Considering the adjacency matrices of the graphs in

Ω'_n, we find that $\xi(n) = \mathrm{card}(M_n)$, where M_n is the set of matrices $((a_{ij}))_{0 \le i,j \le n}$ such that

(i) $\quad a_{ij} \in \{0, c_{ij}\}$ for all i, j,

(ii) $\quad \forall i < n, \exists k : a_{ik} = 1$,

(iii) $\quad \forall j > 0, \exists l : a_{lj} = 1$.

Let $M_{n,m}$ be the set of pairs $\{((a_{ij}))_{0 \le i,j \le n}; \{j_1, \ldots, j_m\}\}$, where $((a_{ij}))_{0 \le i,j \le n}$ is a matrix satisfying (i) and (ii), and where (iii) is invalid when $j \in \{j_1, \ldots, j_m\}$ (and possibly also for other j's).

Now it is easy to see that

$$\mathrm{card}(M_n) = \sum_{i=0}^{n-1} (-1)^i \,\mathrm{card}(M_{n,i}) . \tag{2}$$

We shall use this to find an expression for $\xi(n) = \mathrm{card}(M_n)$.

Definition 3.1. Let H^n_m be the full nth degree homogeneous polynomial in m variables, i.e.,

$$H^n_m(a_1, \ldots, a_m) = \sum a_{i_1} a_{i_2} \cdots a_{i_n} ,$$

summation being over all increasing sequences $1 \le i_1 \le \ldots \le i_n \le m$. Let

$$\pi_n = \prod_{i=1}^{n} (2^i - 1) .$$

Define $\pi_0 = 1, H^0_0 = 0, H^0_i = 1$ if $i > 0$.

One realizes that

$$\mathrm{card}(M_{n,m}) = \sum (2^{i_1} - 1)(2^{i_2} - 1) \ldots (2^{i_n} - 1) , \tag{3}$$

summation being over all sequences such that

$$i_j \le i_{j+1} \le i_j + 1 \quad \text{for all } j < n ,$$

$$i_1 = 1, \qquad i_n = n - m .$$

This is readily seen to be equal to

$$\pi_{n-m} H^m_{n-m}(2^1 - 1, 2^2 - 1, \ldots, 2^{n-m} - 1) .$$

Hence we have proved the following proposition.

Proposition 3.2. *With the above notation,*

$$\xi(n) = \sum_{m=0}^{n-1} (-1)^m \pi_{n-m} H_{n-m}^{m}(2^1 - 1, 2^2 - 1, \ldots, 2^{n-m} - 1).$$

An equivalent form is

$$\xi(n) = \sum_{i=1}^{\infty} (M^{n-1})_{1i} \tag{4}$$

(the sum of the elements in row no. 1 of M^{n-1}), where M is the matrix

$$\begin{bmatrix} -1 & 3 & 0 & 0 & 0 & . & . & . \\ 0 & -3 & 7 & 0 & 0 & . & . & . \\ 0 & 0 & -7 & 15 & 0 & . & . & . \\ 0 & 0 & 0 & -15 & 31 & . & . & . \\ 0 & 0 & 0 & 0 & -31 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}.$$

By (1) and (4) we have

$$\psi(n) = \sum_{i=0}^{n-1} \binom{n-1}{i} \sum_{j=1}^{\infty} (M^{n-1-i})_{1j} = \sum_{j=1}^{\infty} (M+I)_{1j}^{n-1}.$$

Hence

$$\begin{aligned}\psi(n) &= \sum_{m=0}^{n-1} (-1)^m \pi_{n-m} H_{n-m}^{m}(2^1 - 2, 2^2 - 2, \ldots, 2^{n-m} - 2) \\ &= \sum_{m=0}^{n-2} (-1)^m \pi_{n-m} H_{n-1-m}^{m}(2^1 - 1, 2^2 - 1, \ldots, 2^{n-m-1} - 1)2^m.\end{aligned}$$

Thus we have proved the following proposition.

Proposition 3.3. *With the above notation,*

$$\psi(n) = \sum_{m=0}^{n-2} (-2)^m \pi_{n-m} H_{n-m-1}^{m}(2^1 - 1, 2^2 - 1, \ldots, 2^{n-m-1} - 1).$$

4. A recursion formula

Calculation of $\xi(n)$, $\psi(n)$ for small n suggests the following proposition.

Proposition 4.1. $\psi(n) = 2^n \xi(n-1) - 1$.

Proof.

$$2^n \xi(n-1) - \psi(n) =.$$

$$= \sum_{m=0}^{n-2} (-1)^m \, [\pi_{n-m-1} H^m_{n-m-1}(2^1 - 1, \ldots, 2^{n-m-1} - 1) 2^n$$

$$- \pi_{n-m} H^m_{n-m-1}(2^1 - 1, \ldots, 2^{n-m-1} - 1) 2^m]$$

$$= \sum_{m=0}^{n-2} (-1)^m \, [2^m (\pi_{n-m} + \pi_{n-m-1}) - 2^m \pi_{n-m}]$$

$$H^m_{n-m-1}(2^1 - 1, \ldots, 2^{n-m-1} - 1)$$

$$= \sum_{m=0}^{n-2} (-1)^m \pi_{n-m-1} H^m_{n-m-1}(2^2 - 2, \ldots, 2^{n-m} - 2) .$$

It follows that $2^n \xi(n-1) - \psi(n) = \Sigma_{i=1}^{\infty} (N^{n-1})_{1i}$, where N is the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & . & . & . \\ 0 & -2 & 3 & 0 & 0 & . & . & . \\ 0 & 0 & -6 & 7 & 0 & . & . & . \\ 0 & 0 & 0 & -14 & 15 & . & . & . \\ 0 & 0 & 0 & 0 & -30 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix} .$$

Put $A_n = (a_1^n, a_2^n, \ldots) = (1, 0, 0, \ldots) N^{n-1}$, $n \geqslant 1$. We must show that $\Sigma_{i=1}^{\infty} a_i^n = 1$. This is true if $n = 1$. Suppose that it has been proved when $n = k$, and let $A_k = (a_1^k, a_2^k, \ldots)$, only finitely many of the a_i^k's being nonzero. Then $A_{k+1} = A_k N = (0, a_1^k - 2a_2^k, 3a_2^k - 6a_3^k, \ldots)$. We see that only finitely many of the a_i^{k+1}'s are nonzero and that

$$\sum_{i=1}^{\infty} a_i^{k+1} = 0 + a_1^k - 2a_2^k + 3a_2^k - 6a_3^k + 7a_3^k + \ldots = a_1^k + a_2^k + a_3^k + \ldots = 1 .$$

Remark. The simple form of this proposition suggests that one might be able to prove it along the following lines. Let Ω_n'' be the set of directed, connected graphs with vertices $v_{-1}, v_0, \ldots, v_{n-1}$; (v_i, v_j) being an arc only if $i < j$, where each arc not incident to v_{-1} lies along a directed path from v_0 to v_{n-1}. Obviously, $\mathrm{card}(\Omega_n'') = 2^n \xi(n-1) - 1$. Set up a natural $1-1$ correspondence between Ω_n'' and Ω_n. However, we have not been able to find such a proof.

5. Generating functions

In this paragraph we shall generalize Propositions 3.2 and 3.3. We need the following.

Definition 5.1. Let $\Omega_{n,k} = \{H \in \Omega_n : \deg v_0 = k \text{ in } H\}$. $\Omega'_{n,k}$ is defined similarly. Let their cardinalities be $\psi(n, k)$, $\xi(n, k)$.

Proposition 5.2. *Put* $\psi(0,0) = \xi(0,0) = 1$, $\psi(0,k) = \xi(0,k) = 0$, *for* $k > 0$. *Then*

(i) $\psi(n, 0) = \xi(n, 0) = 0$ *for* $n > 0$,

$\psi(n, k) = \xi(n, k) = 0$ *for* $k > n$,

$\psi(1, 1) = \xi(1, 1) = 1$.

For $n \geq 2$, $k \geq 1$,

$$\text{(ii)}\quad \psi(n, k) = \psi(n-1, k) - \psi(n-1, k-1) + \sum_{i=k-1}^{n-1} \psi(n-1, i)\binom{i}{k-1}2^{k-1},$$

$$\text{(iii)}\quad \xi(n, k) = -\xi(n-1, k-1) + \sum_{i=k-1}^{n-1} \xi(n-1, i)\binom{i}{k-1}2^{k-1}.$$

Proof. (i) The proof is clear.

(ii) There are $\psi(n-1, k)$ graphs in $\Omega_{n,k}$ not containing v_1. Let $H \in \Omega_{n,k}$ contain v_1. Define a graph H' by identifying v_0 and v_1. Then $H' \in \Omega_{n-1,i}$ for some $i \geq k-1$.[1] Let $i > k-1$, $K \in \Omega_{n-1,i}$. The number of graphs H such that $H' = K$ is $\binom{i}{k-1}2^{k-1}$, for the number of possible choices of points adjacent to v_0 is $\binom{i}{k-1}$, and for each choice there are 2^{k-1} graphs (any of the $k-1$ points may or may not be adjacent to v_1). On the other

[1] To be pedantic, one should change the names of the vertices in H' from $v_0, v_2, \ldots, v_n$ to $v_0, v_1, \ldots, v_{n-1}$.

hand, if $i = k - 1$, there are $2^{k-1} - 1$ graphs for each choice. We must not count the graphs having no arcs incident out of v_1. (ii) then follows.

(iii) The proof is similar; there are no graphs in $\Omega'_{n,k}$ not containing v_1.

We introduce generating functions.

Definition 5.3. Let $\psi_n : \mathbf{R} \to \mathbf{R}$ be defined by

$$\psi_n(x) = \sum_{k=1}^{n} x^k \psi(n, k), \quad n = 1, 2, \ldots .$$

ξ_n is similarly defined.

Clearly $\psi(n) = \psi_n(1)$, $\xi(n) = \xi_n(1)$.

Proposition 5.4.

$$\text{(i)} \qquad \xi_1(x) = \psi_1(x) = x .$$

For $n > 1$,

$$\text{(ii)} \qquad \psi_n(x) = (1 - x)\, \psi_{n-1}(x) + x\psi_{n-1}(1 + 2x),$$

$$\text{(iii)} \qquad \xi_n(x) = -x\xi_{n-1}(x) + x\xi_{n-1}(1 + 2x) .$$

Proof. (i) The proof is clear.

$$\begin{aligned}
\text{(ii)} \qquad \psi_n(x) &= \sum_{k=1}^{n} x^k \psi(n, k) \\
&= \sum_{k=1}^{n} x^k \psi(n - 1, k) - \sum_{k=1}^{n} x^k \psi(n - 1, k - 1) \\
&\quad + \sum_{k=1}^{n} \sum_{i=k-1}^{n-1} x^k \binom{i}{k-1} 2^{k-1} \psi(n - 1, i) \\
&= \psi_{n-1}(x) - x\psi_{n-1}(x) \\
&\quad + \sum_{i=0}^{n-1} \left(\sum_{k=1}^{i+1} \binom{i}{k-1} 2^{k-1} x^{k-1} \right) x\psi(n - 1, i)
\end{aligned}$$

$$= \psi_{n-1}(x) - x\psi_{n-1}(x) + \sum_{i=0}^{n-1} x(1+2x)^i \psi(n-1, i)$$

$$= (1-x)\,\psi_{n-1}(x) + x\psi_{n-1}(1+2x)\,.$$

(iii) The proof is similar.

We need a definition.

Definition 5.5. Let $y_m : \mathbf{R} \to \mathbf{R}$ be defined by

$$y_m(x) = -1 + 2^{m-1}(1+x), \quad m = 0, 1, 2, \ldots\,.$$

Note that

(5) $\qquad y_m(1+2x) = y_{m+1}(x)\,.$

We have the following theorem.

Theorem 5.6.

$$\xi_n = \sum_{m=0}^{n-1} (-1)^m\, y_1 y_2 \ldots y_{n-m} H^m_{n-m}(y_1, \ldots, y_{n-m})\,.$$

(Notice that $y_m(1) = 2^m - 1$, so Theorem 5.6 is a generalization of Proposition 3.2.)

Proof. The theorem is valid when $n = 1$.

Suppose that it has been proved for $n = k-1$. Then (Proposition 5.4),

$$\xi_k(x) = -x\xi_{k-1}(x) + x\xi_{k-1}(1+2x)$$

$$= -y_1(x)\,\xi_{k-1}(x) + y_1(x)\,\xi_{k-1}(1+2x)$$

$$= -\left\{\sum_{m=0}^{k-2} (-1)^m y_1 \ldots y_{k-m-1} y_1 H^m_{k-m-1}(y_1, \ldots, y_{k-m-1})\right.$$

$$\left. - \sum_{m=0}^{k-2} (-1)^m y_1 \ldots y_{k-m} H^m_{k-m-1}(y_2, \ldots, y_{k-m})\right\}(x) \quad \text{by (5).}$$

But the definition of H^n_m implies that

$$H^m_{k-m}(y_1,\dots,y_{k-m}) = H^m_{k-m-1}(y_2,\dots,y_{k-m}) + y_1 H^{m-1}_{k-m}(y_1,\dots,y_{k-m}).$$

(The last term is 0 when $m = 0$.) Thus, by changing summation index, we find that

$$\begin{aligned}\xi_k &= \sum_{m=0}^{k-1} (-1)^m y_1 \dots y_{k-m} (H^m_{k-m-1}(y_2,\dots,y_{k-m}) \\ &\quad + y_1 H^{m-1}_{k-m}(y_1,\dots,y_{k-m})) \\ &= \sum_{m=0}^{k-1} (-1)^m y_1 \dots y_{k-m} H^m_{k-m}(y_1,\dots,y_{k-m}).\end{aligned}$$

The theorem is then proved.

Let M_y be the matrix

$$\begin{bmatrix} -y_1 & y_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & -y_2 & y_3 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & -y_3 & y_4 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & -y_4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Theorem 5.6 can be written in matrix form

$$(6) \qquad \xi_n = y_1 \sum_{i=1}^{\infty} (M_y^{n-1})_{1i}.$$

We have (similar to (1))

$$(7) \qquad \psi_n = \sum_{i=0}^{n-1} \binom{n-1}{i} \xi_{n-i}.$$

By (6) and (7),

$$\psi_n = y_1 \sum_{i=1}^{\infty} ((M_y + I)^{n-1})_{1i}.$$

This is equivalent to

$$\psi_n = \sum_{m=0}^{n-1} y_1 \dots y_{n-m} H_{n-m}^m (1 - y_1, \dots, 1 - y_{n-m}) .$$

But $1 - y_j = -2y_{j-1}$. Thus we have proved the following.

Theorem 5.7.

$$\psi_n = \sum_{m=0}^{n-1} (-2)^m y_1 \dots y_{n-m} H_{n-m}^m (y_0, \dots, y_{n-m-1}) .$$

It turns out to be fruitful to introduce new quantities η_n, $\eta(n)$, $\eta(n, k)$ defined as follows.

Definition 5.8. Let

$$\eta_n = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \xi_{n-i} . \tag{8}$$

$\eta(n)$, $\eta(n, k)$ are defined similarly.

The definition implies that

$$\xi_n = \sum_{i=0}^{n-1} \binom{n-1}{i} \eta_{n-i}, \qquad \psi_n = \sum_{i=0}^{n-1} \binom{n-1}{i} 2^i \eta_{n-i} . \tag{9}$$

We have, from (6) and (8),

$$\eta_n = y_1 \sum_{i=1}^{\infty} ((M_y - I)^{n-1})_{1i} .$$

Thus we find

$$\eta_n = \sum_{i=0}^{n-1} (-1)^i y_1 \dots y_{n-i} H_{n-i}^i (y_1 + 1, \dots, y_{n-i} + 1) .$$

But $y_i + 1 = 2^{i-1}(1+x)$. So

$$\eta_n = \sum_{i=0}^{n-1} (-1)^i (1+x)^i y_1 \dots y_{n-i} H^i_{n-i}(1, 2, \dots, 2^{n-i-1}) .$$

Now it is easy to prove by induction that

$$H^p_q(1, 2, \dots, 2^{q-1}) = \frac{\pi_{p+q-1}}{\pi_p \pi_{q-1}} .$$

Hence we have proved the following lemma.

Lemma 5.9.

$$\eta_n = \sum_{i=0}^{n-1} (-1)^i (1+x)^i y_1 \dots y_{n-i} \frac{\pi_{n-1}}{\pi_i \pi_{n-i-1}} .$$

Now we can prove a further proposition.

Proposition 5.10. *For $n > 1$,*

$$\eta_n = (1+x)(2^{n-1} - 1)\eta_{n-1} + (-1)^{n-1} x .$$

Proof.

$$\begin{aligned}
\eta_n &= \sum_{i=0}^{n-1} (-1)^i (1+x)^i y_1 \dots y_{n-i} \frac{\pi_{r-1}}{\pi_i \pi_{n-1-i}} \\
&= \sum_{i=0}^{n-2} (-1)^i (1+x)^i y_1 \dots y_{n-i-1} \frac{\pi_{n-1}}{\pi_i \pi_{n-i-1}} (-1 + 2^{n-i-1}(1+x)) \\
&\quad + (-1)^{n-1}(1+x)^{n-1} y_1 \\
&= -\sum_{i=0}^{n-2} (-1)^i (1+x)^i y_1 \dots y_{n-i-1} \frac{\pi_{n-1}}{\pi_i \pi_{n-i-1}} \\
&\quad + \sum_{i=0}^{n-2} (-1)^i (1+x)^{i+1} y_1 \dots y_{n-i-1} \frac{\pi_{n-2}}{\pi_i \pi_{n-2-i}} (2^{n-1} - 1)
\end{aligned}$$

$$+\sum_{i=0}^{n-2} (-1)^i(1+x)^{i+1} y_1 \dots y_{n-i-1} \frac{\pi_{n-1}}{\pi_i \pi_{n-i-1}} + (-1)^{n-1}(1+x)^{n-1} y_1$$

$$= (2^{n-1} - 1)(1+x)\eta_{n-1}$$

$$+ x \sum_{i=0}^{n-1} y_1 \dots y_{n-i-1} H_{n-i}^i(-(1+x), -2(1+x), \dots, -2^{n-i-1}(1+x))$$

$$= (2^{n-1} - 1)(1+x)\eta_{n-1} + x \sum_{i=1}^{\infty} (N_y^{n-1})_{1i},$$

where N_y is the matrix

$$\begin{pmatrix} -(1+x) & x & 0 & 0 & \cdot & \cdot \\ 0 & -2(1+x) & 2(1+x)-1 & 0 & \cdot & \cdot \\ 0 & 0 & -4(1+x) & 4(1+x)-1 & \cdot & \cdot \\ 0 & 0 & 0 & -8(1+x) & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Now we claim that $\Sigma_{i=1}^{\infty} (N_y^{n-1})_{1i} = (-1)^{n-1}$. Indeed, the argument from the proof of Proposition 4.1 applies, so Proposition 5.10 follows.

By means of this proposition it is straight forward to prove the following theorem by induction.

Theorem 5.11.

$$\eta_n = x \sum_{j=0}^{n-1} (-1)^{n-1-j}(1+x)^j \frac{\pi_{n-1}}{\pi_{n-1-j}}.$$

This theorem can be used to find expressions for $\eta(n, k)$ and thus (by means of (9)) for $\xi(n, k)$ and $\psi(n, k)$. We then find the following proposition.

Proposition 5.12.

(i) $$\eta(n,k)=\pi_{n-1}\sum_{j=0}^{n-k}(-1)^j\binom{n-1-j}{k-1}\frac{1}{\pi_j},$$

(ii) $$\xi(n,k)=\sum_{i=0}^{n-k}\sum_{j=0}^{n-k-i}(-1)^j\binom{n-1}{i}\binom{n-1-i-j}{k-1}\frac{\pi_{n-1-i}}{\pi_j},$$

(iii) $$\psi(n,k)=\sum_{i=0}^{n-k}\sum_{j=0}^{n-k-i}(-1)^j 2^i\binom{n-1}{i}\binom{n-1-i-j}{k-1}\frac{\pi_{n-1-i}}{\pi_j}.$$

This gives the following.

Corollary 5.13. *For fixed k, all the three functions $\eta(n,n-k)$, $\xi(n,n-k)$, $\psi(n,n-k)$ are asymptotically equal to*

$$C\cdot 2^{\binom{n}{2}}\binom{n-1}{k},$$

where $C=\prod_{i=1}^{\infty}(1-2^{-i})\simeq 0{,}2888$.

One can also prove that, for fixed k, all the three functions $\eta(n,k)$, $\xi(n,k)$, $\psi(n,k)$ are asymptotically equal to

$$C^2\cdot 2^{\binom{n}{2}}\binom{n-1}{k},$$

where C is as above. We leave this to the reader.

Further consequences of Theorem 5.11 are:

Proposition 5.14.

(i) $$\xi_n=x\sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i}(-1)^{n-1-i-j}\binom{n-1}{i}(1+x)^j\frac{\pi_{n-1-i}}{\pi_{n-1-i-j}},$$

(ii) $$\psi_n=x\sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i}(-1)^{n-1-i-j}2^i\binom{n-1}{i}(1+x)^j\frac{\pi_{n-1-i}}{\pi_{n-1-i-j}}.$$

These may be used instead of Theorems 5.6 and 5.7. Putting $x=1$ we also get new expressions for $\xi(n)$ and $\psi(n)$.

Note that $\eta_n(-1) = (-1)^n$. From this (or from Proposition 5.14) we find that $\psi_n(-1) = -1$, $n \geq 1$; $\xi_n(-1) = 0$, $n \geq 2$. This has a geometric interpretation. We need the following.

Definition 5.15. Let

$$\Omega_n^o(\Omega_n^e) = \{H \in \Omega_n\colon \deg v_0 \text{ is odd (even) in } H\}\ .$$

$\Omega_n'^o$, $\Omega_n'^e$ are defined similarly. Let their cardinalities be $\psi^o(n)$, $\psi^e(n)$, $\xi^o(n)$, $\xi^e(n)$.

Proposition 5.16.

(i) $\quad \psi^o(n) - \psi^e(n) = 1, \quad n = 1, 2, \dots,$

(ii) $\quad \xi^o(n) - \xi^e(n) = 0, \quad n = 2, 3, \dots.$

Proof. Clearly $\psi^o(n) - \psi^e(n) = -\psi_n(-1)$; $\xi^o(n) - \xi^e(n) = -\xi_n(-1)$.

Problem. Is there any interpretation of $\eta(n, k)$ as the cardinality of some natural subset of $\Omega'_{n,k}$?

6. Tables

Table 1. $\eta(n, k)$.

k \ n	1	2	3	4	5	6	7
1	1	0	1	6	91	2820	177661
2		1	3	28	510	18631	1351413
3			3	42	1050	48360	4220433
4				21	945	61845	6942915
5					315	39060	6357015
6						9765	3075975
7							615195
$\eta(n)$	1	1	7	97	2911	180481	22740607

Table 2. $\xi(n, k)$.

$k \backslash n$	1	2	3	4	5	6	7
1	1	1	2	10	122	3346	196082
2		1	5	40	644	21496	1471460
3			3	51	1236	54060	4527228
4				21	1029	66780	7328580
5					315	40635	6596100
6						9765	3134565
7							615195
$\xi(n)$	1	2	10	122	3346	196082	23869210

Table 3. $\psi(n, k)$.

$k \backslash n$	1	2	3	4	5	6	7
1	1	2	5	20	179	4082	218225
2		1	7	58	838	25171	1610977
3			3	60	1458	60780	4[illegible]71193
4				21	1113	72135	7745115
5					315	42210	6844635
6						9765	3193155
7							615195
$\psi(n)$	1	3	15	159	3903	214143	25098495

Reference

[1] E. Andresen and K. Kjeldsen, Weight and multigram distribution of some compound sequences, to appear.