Vortex Equations on Riemannian Surfaces

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Introduction

- Yang Mills equations: originated from electromagnetism and field theory.
- Maxwell's equations are a trivial case of the Yang Mills equations.

Introduction

- The Yang Mills equation looks like $d_A^* F_A = 0$ where F_A is the curvature induced by a “connection” of a base space
- To obtain Maxwell’s equations on spacetime (standard $\mathbb{R}^4 = (x, y, z, t)$), we have

$$\begin{aligned} F_A = {} & E_x dy \wedge dz + E_y dz \wedge dx + E_z dz \wedge dx + B_x dx \wedge dt \\ & + B_y dy \wedge dt + B_z dz \wedge dt. \end{aligned}$$

- d_A^* in this case is $*d*$, where $*$ is an operation on $\mathbb{R}^4$ called “Hodge star”.

Introduction

$$
\begin{aligned}
d_A^* F_A &= d * (E_x dy \wedge dz + \ldots + B_x dx \wedge dt) \\
&= *d(E_x dx \wedge dt + \ldots + B_x dy \wedge dz) \\
&= *(\frac{\partial E_x}{\partial x} dx + \frac{\partial E_x}{\partial y} dy + \frac{\partial E_x}{\partial z} dz + \frac{\partial E_x}{\partial t} dt) dx \wedge dt + \ldots \\
&+ (\frac{\partial B_x}{\partial x} dx + \frac{\partial B_x}{\partial y} dy + \frac{\partial B_x}{\partial z} dz + \frac{\partial B_x}{\partial t} dt) dy \wedge dz + \ldots
\end{aligned}
$$

Introduction

$$
\begin{aligned}
&= *((\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t})dy \wedge dz \wedge dt + ... \\
&+ (\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z})dx \wedge dy \wedge dz) \\
&= (\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t})dx + ... + (\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z})dt \\
&= 0
\end{aligned}
$$

This implies the Maxwell equations.

Introduction

Background topics to be briefly covered:

- Lie groups
- Bundles, connection and curvature
- Hodge operator
- Symplectic manifolds with Hamiltonian action

Lie groups

- Recall a *smooth manifold* is a topological space which can be covered by open charts $(\mathcal{U}_\alpha, \phi_\alpha)$, where ϕ_α is a bijective map from an open subset of $\mathbb{R}^n$ to $\mathcal{U}_\alpha$ and such that the transition maps between overlapping charts are smooth.
- A *Lie group* G is a group and a smooth manifold such that group multiplication $\mu : G \times G \to G$ is smooth.
- Denote by $L_g : G \to G$ and $R_g : G \to G$ the left and right multiplication by an element $g \in G$.

Lie groups

Some important Lie groups include

- the circle $S^1 \subset \mathbb{C}$ with the induced multiplication operation from $\mathbb{C}$.
- a vector space V with the addition operation.
- the set of nondegenerate $n \times n$ matrices $GL_n \subset Mat_n$ with the induced multiplication operation

Lie groups

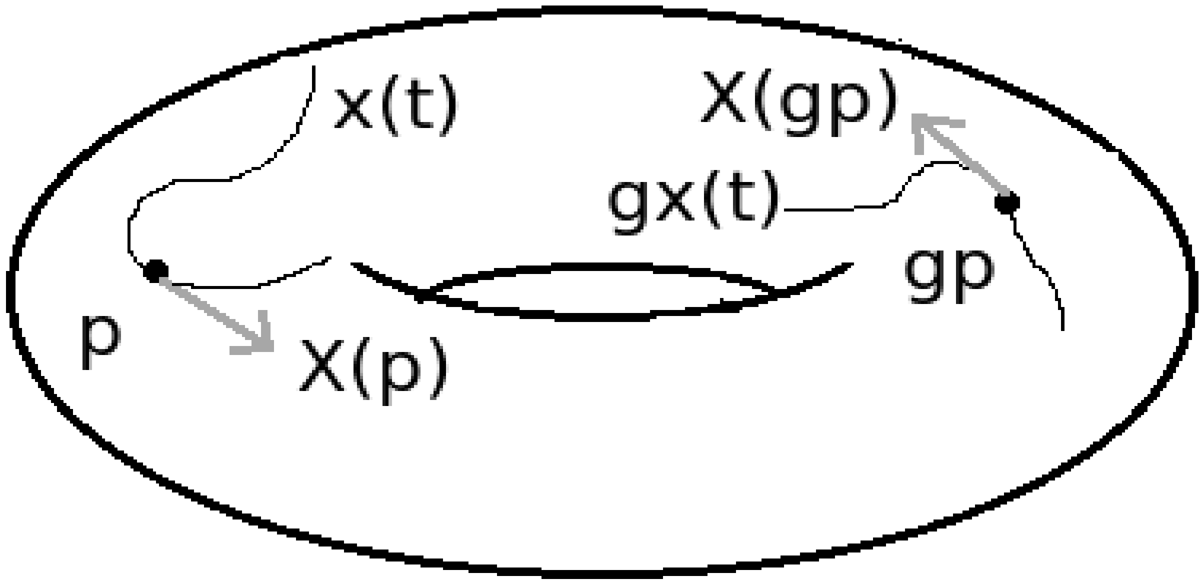
- Recall that a tangent vector v at $p \in M$ is the derivative $\dot{\gamma}(0)$ at $\gamma(0) = p$ of a smooth curve $\gamma(t)$
- The tangent space T_pM of M at p is the set of all tangent vectors at p. Note that T_pM is a vector space.

Lie groups

- Given a vector $v \in T_eG$, we can obtain $dL_g(v)$ a vector in T_gG for $g \in G$
- Thus we obtain a vector field X_v called the *left invariant vector field generated by* v
- $T_eG \Leftrightarrow \{$left invariant vector fields on G $\}$
- The Lie bracket on vector fields induces a Lie bracket on T_eG making T_eG a Lie algebra.
- From now on, denote $\mathfrak{g} = T_eG$

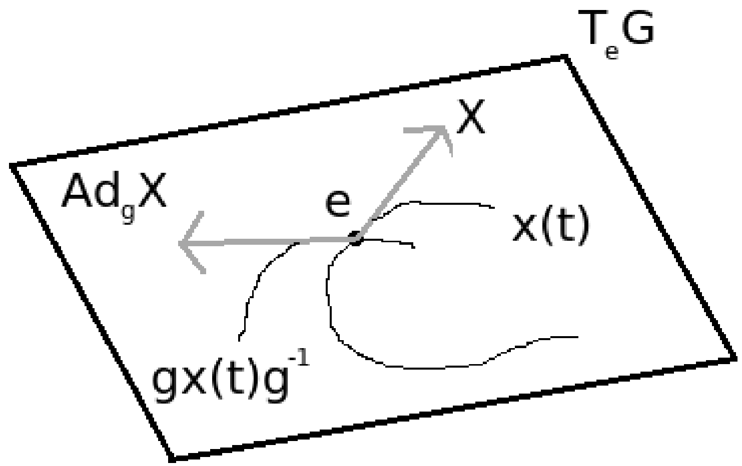
.

Left invariant



Lie groups

- For each $g \in G$, we can define the conjugate map $G \to G$: $h \mapsto ghg^{-1}$.
- We can hence define the *adjoint action* $Ad_g : \mathfrak{g} \to \mathfrak{g}$



Lie groups

- Given $v \in \mathfrak{g}$ and the corresponding left invariant vector field X_v, we can consider the unique local flow at e.
- Denote this local flow by $\exp tv(e)$ where $t \in (\epsilon, \epsilon)$ and $\exp(0v)(e) = e$
- It satisfies $\frac{d}{dt}|_{t=0}(\exp tv(e)) = v$

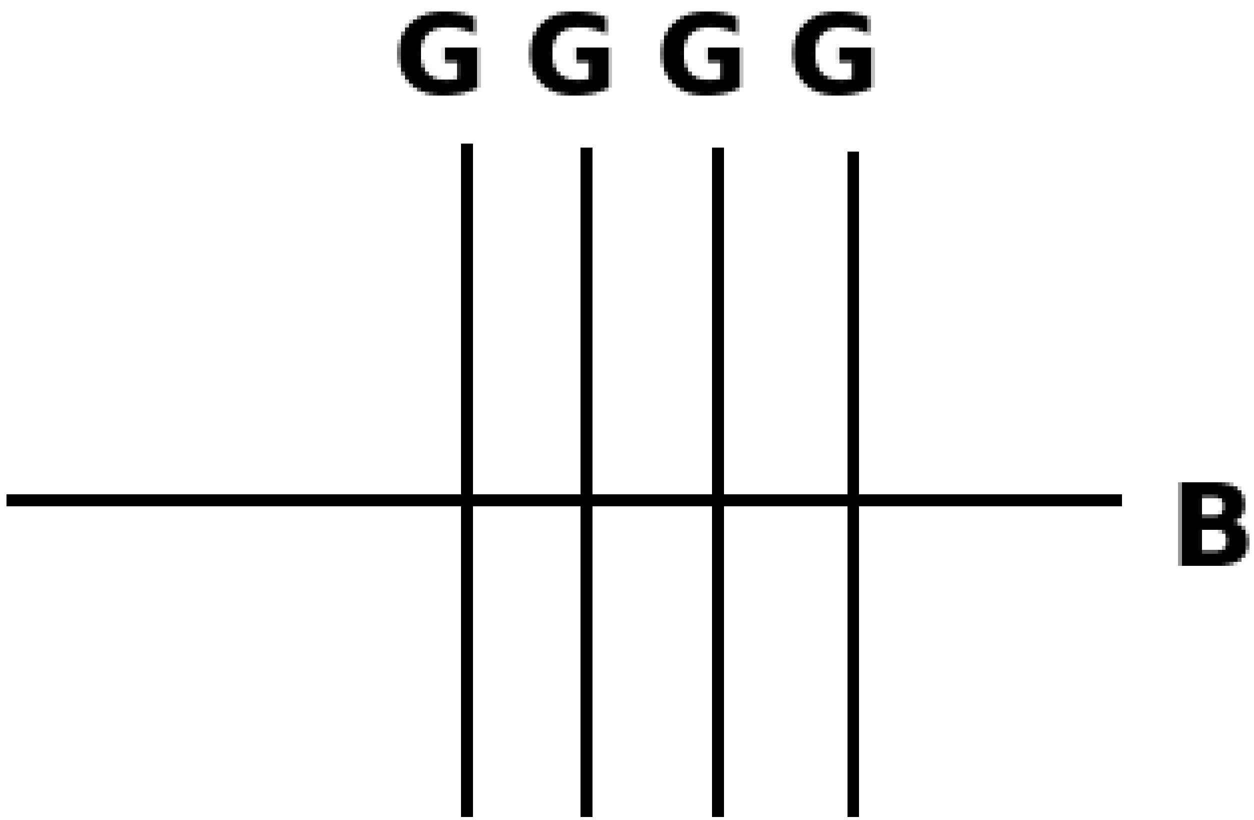
Principal Bundles

A **principal G-bundle,** $\pi : P \to B$, consists of

- a total space P
- a base space B
- a continuous projection map $\pi : P \to B$
- a right action of the group G on P

For $b \in B$, $\pi^{-1}(b)$ is called the fiber over b. Each fiber is homeomorphic to G, and G acts within fibers.

Principal Bundles

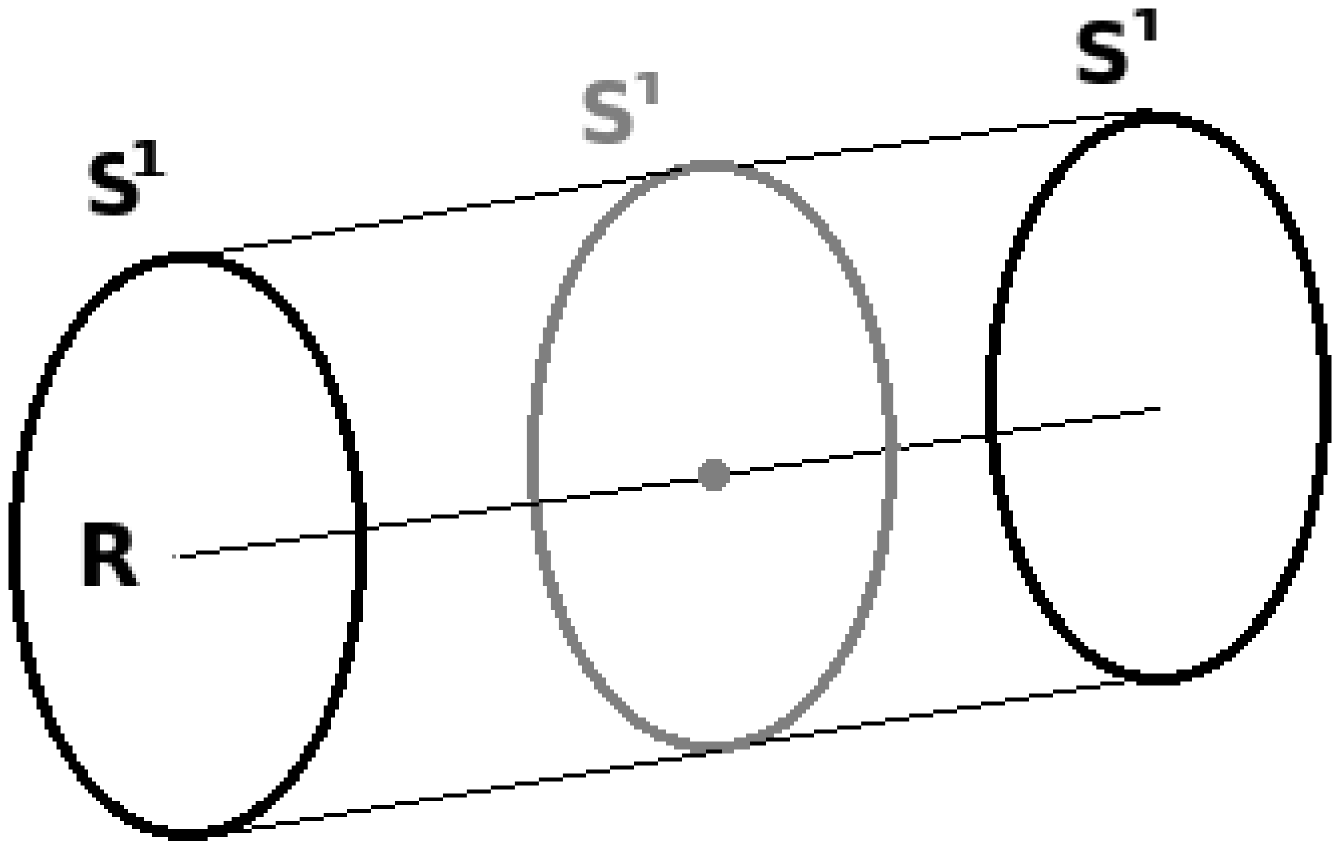


Example: Principal S^1-Bundle

- $G = S^1$ (circle group)
- $B = \mathbb{R}$
- $P = \mathbb{R} \times S^1$ (a cylinder of radius 1)
- π the projection onto the first factor
- The action of G given by multiplication on the second factor

Then each fiber looks like S^1, and the group S^1 acts within the fibers.

Example: Principal S^1-Bundle



Vector Bundle

A **vector bundle** $\pi : E \to B$:

- similar to principal G-bundle
- no group or group action
- each fiber has the structure of a vector space

The tangent bundle of a 2-sphere is a vector bundle we can visualize.

Example: The Adjoint Bundle

We need:

- A principal G-bundle
- The Lie algebra $\mathfrak{g}$ of the group G
- The adjoint action of G on $\mathfrak{g}$
- The equivalence relation $(p, X) \sim (pg, \mathrm{Ad}_{g^{-1}} X)$

Then the adjoint bundle is given by $P(\mathfrak{g}) = (P \times \mathfrak{g})/\sim$.
Fibers looks like the vector space $\mathfrak{g}$.

Connections

- On a principal G-bundle P, a connection assigns a horizontal subspace H_p to each tangent space T_pP of P
- For computation: connection form $A \in \Omega^1(P, \mathfrak{g})$
- $\ker A$ is horizontal
- A is vertical: $A(X^\#) = X$ for $X \in \mathfrak{g}$
- Write: $A = \sum A_i \otimes e_i$
- On vector bundles: connection $\Leftrightarrow$ covariant derivative

Example: Connection

- $G = \mathbb{R}$
- $B = \mathbb{R}$
- $P = \mathbb{R} \times \mathbb{R}$
- π the projection on the first factor
- G-action: group addition on second factor
- $\{x, y\}$ standard basis for $P \Rightarrow \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ basis for $T_p P$
- $A = dy \Leftrightarrow x$ direction is horizontal and y direction is vertical

Curvature

We need:

- principal G-bundle $\pi : P \to B$
- connection A on P

Then

- $dA \in \Omega^2(P, \mathfrak{g})$
- Curvature form $F_A \in \Omega^2(B, P(\mathfrak{g}))$
- Related by structure equation:

$$\pi^* F_A = dA|_{horiz} = dA + \frac{1}{2}[A, A]$$

Example: Curvature

Recall the example of a principal $\mathbb{R}$-bundle $\mathbb{R} \times \mathbb{R}$ with π the projection on the first factor, and the connection form $A = dy$.

Then $dA = d^2y = 0$, and so the curvature of this connection is zero. We say such a connection is flat.

Hodge operator

- Given an orientable Riemannian manifold (Σ, g).
- For each tangent space $T_x\Sigma$, we can define the linear operator $* : \Lambda^{\cdot}(T_x^*\Sigma) \to \Lambda^{n-\cdot}(T_x^*\Sigma)$ such that for any positively oriented orthonormal basis $\{e_1, ..., e_n\}$.
  $*(1) = e_1^* \wedge ... \wedge e_n^*$
  $*(e_1^* \wedge ... \wedge e_k^*) = e_{k+1}^* \wedge ... \wedge e_n^*$
  $...$
  $*(e_1^* \wedge ... \wedge e_n^*) = 1$

Hodge operator

- We can thus define $* : \Omega^k(\Sigma) \to \Omega^{n-k}(\Sigma)$.
- $*$ is called the *Hodge operator*.
- If Σ is compact, one can define an inner product $<,>: \Omega^k(\Sigma) \times \Omega^k(\Sigma) \to \mathbb{R}$ by

$$< \alpha, \beta >= \int_\Sigma \alpha \wedge *\beta.$$

- Notice that $< *\alpha, *\beta >=< \alpha, \beta >$.

Symplectic manifold

- A *symplectic manifold* M is a smooth manifold with a nondegenerate closed 2-form ω.
- A *symplectomorphism* ϕ on M is a diffeomorphism on M and preserves the form $\phi^*\omega = \omega$.
- A vector field v on M is *Hamiltonian* if there exists a function $H : M \to \mathbb{R}$ such that $i_v(\omega) = dH$.

Symplectic manifold

A (right) G-action of symplectomorphisms on (M, ω) is *Hamiltonian* if

- For every $v \in \mathfrak{g}$, the vector field $X_v^\sharp$ is a Hamiltonian vector field. This is equivalent to the existence of $\Phi : M \to \mathfrak{g}^*$ such that

  $$i_{X_\xi^\sharp} = d\Phi^\xi$$

  where $\Phi^\xi(x) = \Phi(x)(\xi)$.
- Φ is equivariant, i.e.

  $$\Phi \circ g = Ad^*_{g^{-1}} \circ \Phi.$$

Vortex equations

- Let G be a compact Lie group and $\mathfrak{g}$ is equipped with an invariant metric $<,>$, i.e.

$$< Ad_g v_1, Ad_g v_2 >=< v_1, v_2 > \forall v, w \in \mathfrak{g}, \forall g \in G.$$

- Let Σ be a Riemannian surface, P a principal G-bundle over the base Σ and (M, ω, G, Φ) a symplectic manifold with a Hamiltonian action.
- For any equivariant $u : P \to M$, we obtain a section of $u : \Sigma \to P \times_G M$, where $P \times_G M = P \times G / \{(p, m) \sim (pg, g^{-1}m)\}, \ \forall g \in G$.

Vortex equations

- Define $F : \mathcal{A}(P) \times \Gamma(\Sigma, P \times_G M) \to \Omega^2(\Sigma, \mathcal{P}(\mathfrak{g}))$ by

$$F(A, u) = F_{A,u} = F_A + u^*\Phi \mathrm{dVol}_\Sigma.$$

- Consider the functional

$$f(A, u) = \|F_{A,u}\|^2_{L^2} = \int_\Sigma < F_{A,u} \wedge *F_{A,u} >$$

- (A, u) satisfies the *vortex equation* if it is a critical point of f

Vortex equations

- Denote by $\mathcal{M} = \mathcal{A}(P) \times \Gamma(\Sigma, P \times_G M)$.
- The $T_{(A,u)}\mathcal{M} = \Omega^1(\Sigma, \mathcal{P}(\mathfrak{g})) \times \Gamma(\Sigma, \text{Vert}(u^* T(P \times_G M))$
- Then (A, u) is a critical point if and only if for every variation of section of $u_t \in \Gamma(\Sigma, P \times_G M)$ and $a \in \Omega^1(\Sigma, \mathcal{P}(\mathfrak{g}))$,

$$\begin{aligned} \frac{d}{dt}|_{t=0} \|F_{A+t\pi^* a, u_t}\|^2_{L^2} &= \frac{d}{dt}|_{t=0} \int_\Sigma < F_{A+t\pi^* a, u_t} \wedge *F_{A+t\pi^* a, u_t} > \\ &= 0. \end{aligned}$$

Sketch of the computation

- To see that, we need a few ingredients.
- Let $v = \frac{d}{dt}|_{t=0} u_t$.
- Notice $\frac{d}{dt}|_{t=0} F_{A+t\pi^* a} = \pi_*(d\pi^* a + [A \wedge \pi^* a]) = \pi_*(d_A a)$ by definition of d_A.
- Using the property of Hodge operator, the fact that $<,>$ on $\mathfrak{g}$ is invariant and Stokes theorem, we can show that

$$\int_\Sigma < d_A a \wedge * F_A >= \int_\Sigma < a \wedge d_A * F_A >$$

Sketch of the computation

- The symplectomorphism of the group action on M induces a symplectic 2-form ω on Vert$T(P \times_G M)$ from the form ω of M.
- Using the Hamiltonian property of Φ, we obtain $d\Phi_{u(x)}(v) = -i_v \omega_{u(x)}$.
- Thus

$$\int_\Sigma < \frac{d}{dt}|_{t=0} u_t^* \Phi \mathrm{dVol}_\Sigma \wedge *F_{A,u} > = \int_\Sigma < u^* d\Phi(v), *F_{A,u} > \mathrm{dVol}_\Sigma$$
$$= \int_\Sigma \omega_{u(x)}(X^\sharp_{*F_{A,u}}, v) \mathrm{dVol}_\Sigma$$

Sketch of the computation

- We can write the condition of critical point as

$$2\int_\Sigma < a \wedge *F_{A,u} > +2\int_\Sigma \omega_{u(x)}(X^{\#}_{*F_{A,u}}, v)\text{dVol}_\Sigma = 0$$

- Hence, by nondegeneracy of ω, we obtain

$$d_A^* F_{A,u} = *d_A * F_{A,u} = 0$$

and

$$X^{\sharp}_{*F_{A,u}} = 0.$$

Regularity of solutions

- One may ask what the regularity of such solutions are, i.e. are they in certain familiar spaces.
- One way is to start from weak solution and use some regularity theory in PDE to bootstrap the regularity of the solution.
- One may not expect such solutions are smooth. A simple counterexample we understood is the trivial bundle with S_1 action where there are solutions in C^1 but not of higher derivative class.

Regularity of solutions

- However, it may be that under some transformation, the solution can become smooth.
- A *gauge transformation* $\psi : P \to P$ is a map satisfying $\psi(pg) = \psi(p)g$ and $\pi \circ \psi =$ Id on Σ.
- Under the gauge transformation defined on $\mathcal{M}$ by natural pull-back operations, we obtain another solution.

Regularity of solutions

- **Conjecture:** Any solution to the vortex equation can be gauge transformed to a smooth solution.
- We are still working on this conjecture.

References

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