Pebbling on Cartesian Product Graphs

Ryan *W*ard[1] Steven *W*u[2]

[1]Bucknell University

[2]Bard College

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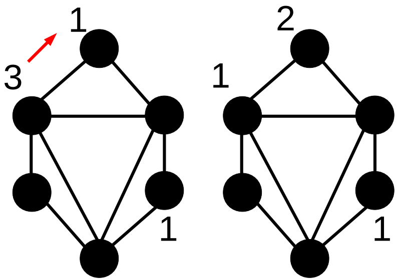
Introduction to Pebbling

Definition

A *configuration* of pebbles is a distribution of pebbles onto the vertices of a graph G.

Definition

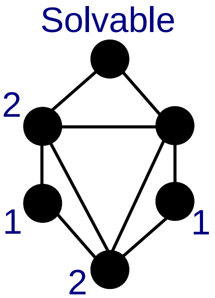
A *pebbling move* consists of removing two pebbles at a vertex and placing one pebble at an adjacent vertex.

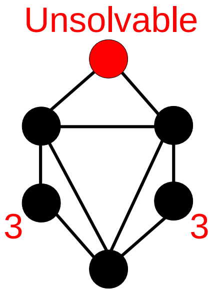


Solvable Configurations

Definition

A configuration is *solvable* if every vertex could be reached by a pebble after a series of pebbling moves.





Pebbling Number

Definition

The *pebbling number* of a graph, denoted $\pi(G)$, is the number of pebbles needed to guarantee that all configurations of $\pi(G)$ pebbles are solvable. We call a graph *class-0* if $|V(G)| = \pi(G)$.

Goal of Project: Determine or set bounds on the pebbling number of certain graphs.

Cartesian Product Graph

Definition

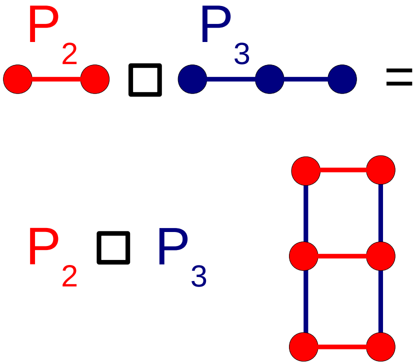
The *Cartesian product* $G \Box H$ is the graph given by

$$V(G \Box H) = \{(g, h) : g \in V(G), h \in V(H)\}$$

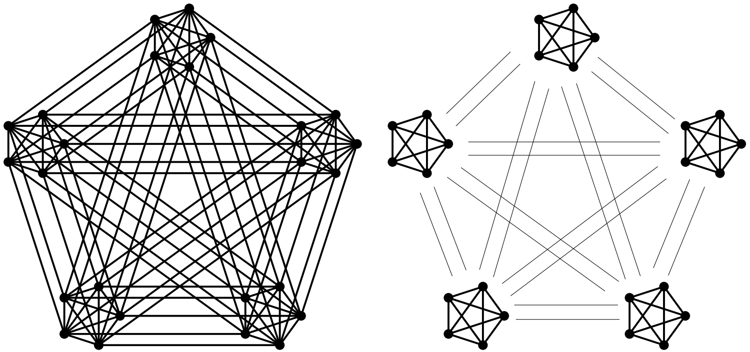
and

$$E(G \Box H) = \{\{(g_1, h_1), (g_2, h_2)\} : [g_1 = g_2 \text{ and } \{h_1, h_2\} \in E(H)]$$
$$\text{or } [\{g_1, g_2\} \in E(G) \text{ and } h_1 = h_2]\}$$

Example: $P_2 \square P_3$



Example: $K_5 \square K_5$



Graham's Conjecture

Conjecture (Graham)

$\pi(G \square H) \leq \pi(G)\pi(H)$

Known for some cases:

- tree with a tree
- cycle with a cycle
- products of paths

Implied by Graham's: If G is class-0, then $G \square G$ is class-0.

What kind of Graph Product is class-0

Can we have a graph G that is not class-0, but the Cartesian product $G \Box G$ is class-0?

A possible answer

The family of graphs with diameter 2 and 2-connectivity but not class-0.

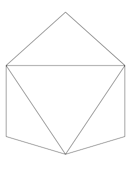


Figure: The simplest graph of the family

An Alternate Problem: Graham's Conjecture

Does the inequality $\pi(G \square G) \leq \pi(G) \times \pi(G)$ hold for this graph?

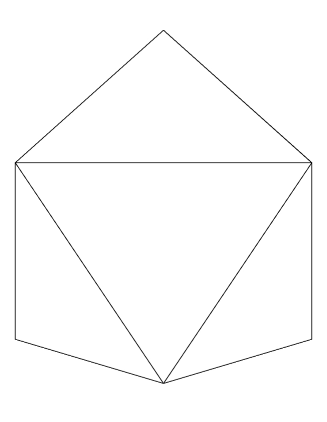
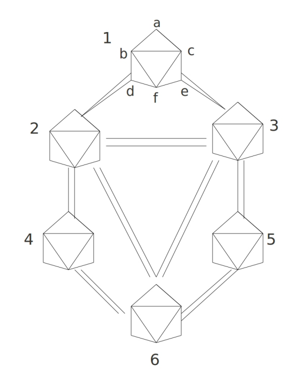


Figure: $\pi(G) = 7, \pi(G) \times \pi(G) = 49$

The product of the graph

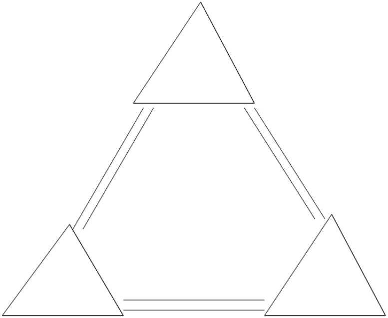
An Alternate Problem

Can we pebble every vertex on this graph with any distribution of 49 pebbles on the vertices?



A class-0 subgraph within the product graph

A simplified $K_3 \square K_3$



A class-0 subgraph within the product graph

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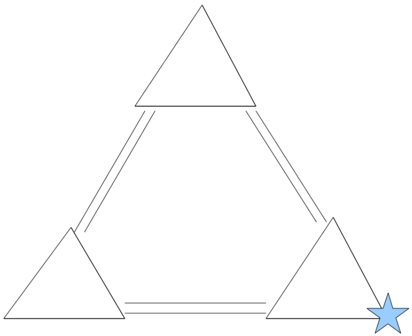
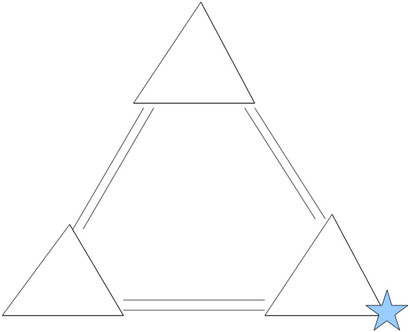


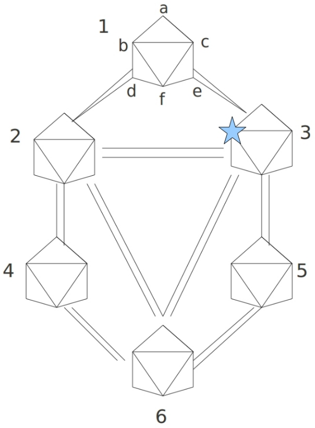
A class-0 subgraph within the product graph

A simplified $K_3 \square K_3$

The other vertices have less than 9 pebbles

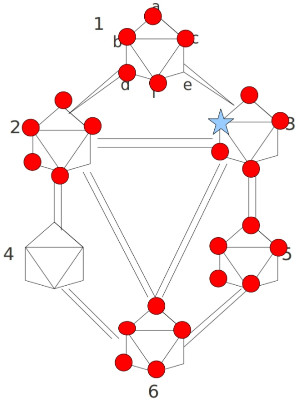


Select the target vertex



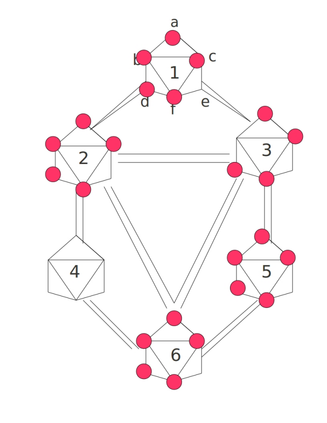
The red vertices are in the same $K_3 \square K_3$

The sum of the pebbles on the red vertices is no more than 24.
The sum of the pebbles on the rest of the 11 vertices is at least 12.



Consider the following case

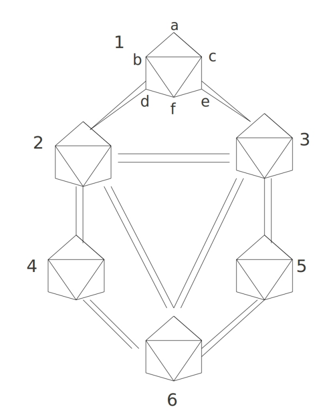
Each of the red vertices has exactly one pebble, and the sum of their pebbles reaches the maximum 24. Since there are 12 pebbles left on the remaining 11 vertices, by Pigeonhole Principle we can move at least one pebble to any of the $K_3 \square K_3$, in which we have enough pebbles to reach the target vertices.



More cases and details to work on

Three kinds of vertices

- Vertex with degree 4. Each in 1 copy of $K_3 \square K_3$.
- Vertex with degree 6. Each in 3 copies of $K_3 \square K_3$.
- Vertex with degree 8. Each in 9 copies of $K_3 \square K_3$.



Open Problems Following

Generalize

- Can we show that other graphs of diameter 2 and 2-connectivity also satisfy Graham's conjecture?
- Can we show that the all the products of such graphs are class-0?
- Can we construct similar graphs with similar symmetrical structure and complete subgraphs?

Acknowledgement

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