Antimagic Labellings of Graphs

DIMACS REU Student Presentation 1

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In this presentation, we will take graphs to be finite, simple, and connected.

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In a magic labelling, the edges of the graph are labelled distinctly from 1 to $|E|$ such that the sum of the edges at each vertex is the same.

Example of Magic Labelling

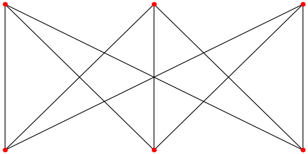

Figure: $K_{3,3}$: The complete bipartite graph on 3 and 3 vertices

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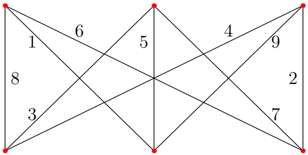

Figure: A magic labelling of $K_{3,3}$

Each vertex sum is equal to 15.

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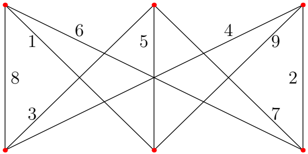

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This is related to the magic square: the top vertices correspond to the rows and the bottom vertices to the columns.

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More precisely, for a graph (V, E), an antimagic labelling is a bijection

$$f : E \to \{1, 2, \ldots, |E|\}$$

such that, for any two distinct vertices $v_1, v_2 \in V$, we have

$$\sum_{\substack{e_i \in E \\ \text{incident to } v_1}} f(e_i) \neq \sum_{\substack{e_j \in E \\ \text{incident to } v_2}} f(e_j)$$

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We call a graph antimagic if it admits an antimagic labelling.

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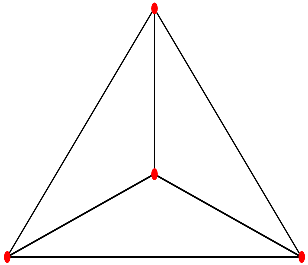

Figure: K_4: The complete graph on 4 vertices

Example Antimagic Labelling

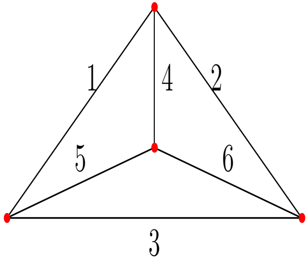

Figure: A labelling of K_4...

Example Antimagic Labelling

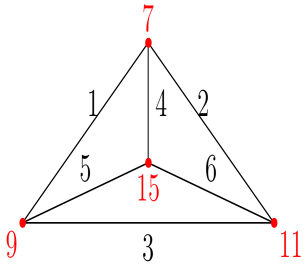

Figure: A labelling of K_4 which is antimagic

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- $C_m \times C_n$ (Wang 2005), $C_m \times P_n$, $P_m \times P_n$ (Cheng 2006)

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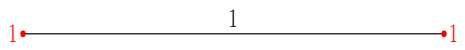

Figure: The only labelling of K_2

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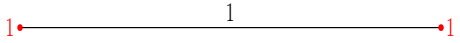

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- Any others?

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Conjecture

All finite, simple, connected graphs except K_2 are antimagic. (Hartsfield and Ringel, 1990)

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I am working towards proving this conjecture for regular graphs with perfect matchings.

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It is conjectured that the generalized Petersen graphs $P(n, k)$ are (a, d)-antimagic for some a and d (Miller and Bača 2000).

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For a given graph, which constants can be achieved?
For what graphs can all constants be achieved?