

Graph Labellings

DIMACS REU Final Presentation

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Graph Labellings

Definition

A **labelling** of a graph is an assignment of numbers, set elements, or group elements to the vertices or edges of the graph that satisfies some properties.

In this presentation, we will take graphs to be finite, simple, and connected.

Antimagic Labelling

An antimagic labelling of a graph G is a labelling of the edges of a graph with the integers 1 to $|E(G)|$ such that every vertex has a different sum of the labels of the incident edges.

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More precisely, for a graph (V, E) , an antimagic labelling is a bijection

$$f : E \rightarrow \{1, 2, \dots, |E|\}$$

such that, for any two distinct vertices $v_1, v_2 \in V$, we have

$$\sum_{\substack{e_i \in E \\ \text{incident to } v_1}} f(e_i) \neq \sum_{\substack{e_j \in E \\ \text{incident to } v_2}} f(e_j)$$

We call a graph **antimagic** if it admits an antimagic labelling.

Example Antimagic Labelling

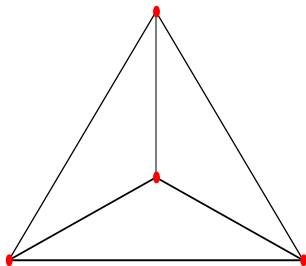


Figure: K_4 : The complete graph on 4 vertices

Example Antimagic Labelling

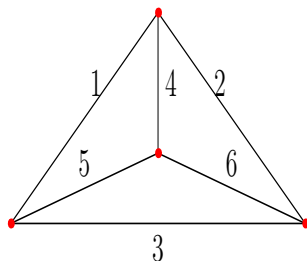


Figure: A labelling of K_4 ...

Example Antimagic Labelling

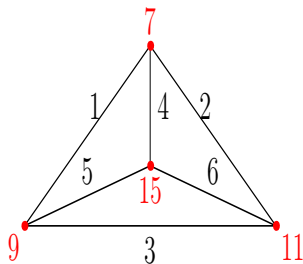
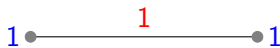


Figure: A labelling of K_4 which is antimagic

What graphs aren't antimagic?

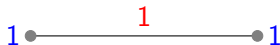
► K_2



The only labelling of K_2

What graphs aren't antimagic?

► K_2



The only labelling of K_2

► Any others?

What graphs are antimagic?

Conjecture

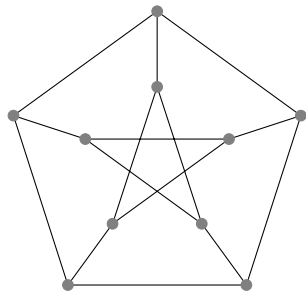
*All finite, simple, connected graphs except K_2 are antimagic.
(Hartsfield and Ringel, 1990)*

My research with antimagic labellings has been towards proving this conjecture, specifically for regular graphs.

Generalized Petersen Graphs

Definition

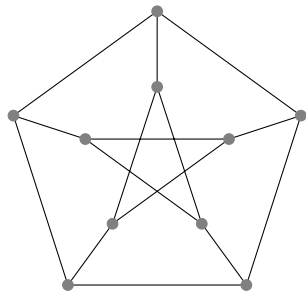
The **Petersen Graph** is the graph illustrated below:



Generalized Petersen Graphs

Definition

The **Petersen Graph** is the graph illustrated below:



Alternatively, we can define the Petersen Graph as the graph with vertex set $V = \{x_0, \dots, x_4, y_0, \dots, y_4\}$ and edge set $E = \{x_i y_i, x_i x_{i+1}, y_i y_{i+2} \mid i = 0, \dots, 4\}$ (where indices are read modulo 5).

Generalized Petersen Graphs

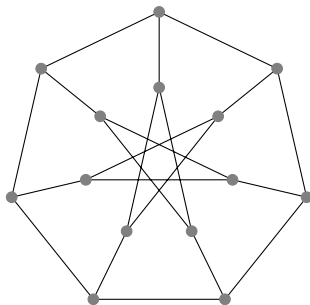
Definition

The **Generalized Petersen Graph** $P(n, k)$ is the graph with vertex set $V = \{x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}\}$ and edge set $E = \{x_i y_i, x_i x_{i+1}, y_i y_{i+k} \mid i = 0, \dots, n-1\}$ (where indices are read modulo k).

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The Generalized Petersen Graph $P(7, 3)$

Generalized Petersen Graphs

Theorem

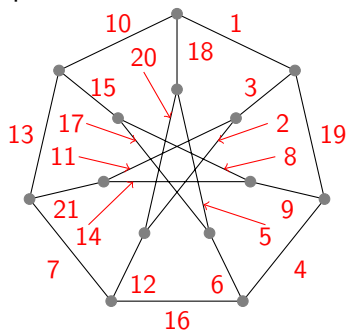
Generalized Petersen Graphs are antimagic.

Generalized Petersen Graphs

Theorem

Generalized Petersen Graphs are antimagic.

Example:



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Assume that H contains an n -cycle such that no two vertices in the n -cycle are connected in M .

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Assume additionally that all of the remaining cycles in H have lengths of the same parity.

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Then G is antimagic.

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Let $G = H \cup M$ be a 3-regular graph with $2n$ edges and a perfect matching M .

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Assume additionally that all of the remaining cycles in H have lengths of the same parity.

Then G is antimagic.

I would like to extend this work to loosen the above conditions and see whether all 3-regular graphs with perfect matchings are antimagic.

$\mathbb{Z}_k$ -magic labellings

Definition

A $\mathbb{Z}_k$ -magic labelling is an assignment of nonzero elements of the group $\mathbb{Z}_k$ to the edges of a graph such that the sum at each vertex (where the sum is taken in $\mathbb{Z}_k$) is the same.

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For a $\mathbb{Z}_k$ -magic labelling of the graph G , define the **index** as the sum of the edge labels at each vertex. Define the **index set** $I_k(G)$ as the set of all indices which can be obtained.

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For a $\mathbb{Z}_k$ -magic labelling of the graph G , define the **index** as the sum of the edge labels at each vertex. Define the **index set** $I_k(G)$ as the set of all indices which can be obtained.

For what graphs and what k do we have that $I_k(G) = \mathbb{Z}_k$?

$\mathbb{Z}_k$ -magic labellings

Theorem

Let G be a regular graph. If G admits a perfect matching, then $I_k(G) = \mathbb{Z}_k$ for all $k \geq 3$. (Lin and Wang 2009)

$\mathbb{Z}_k$ -magic labellings

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We ask whether the above implication can be reversed. If a graph G is regular and $I_k(G) = \mathbb{Z}_k$ for all $k \geq 3$, then does G admit a perfect matching?

$\mathbb{Z}_k$ -magic labellings of cubic graphs

Theorem

Let G be a 3-regular graph. If $I_k(G) = \mathbb{Z}_k$ for all $k \geq 3$, then G admits a perfect matching.

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Theorem

Let G be a 3-regular graph. If $I_k(G) = \mathbb{Z}_k$ for all $k \geq 3$, then G admits a perfect matching.

Proof.

Assume that G has a $\mathbb{Z}_4$ -magic labelling with index 0. Then, for any vertex, there must be an even number of odd labels incident to that vertex.

$\mathbb{Z}_k$ -magic labellings of cubic graphs

Theorem

Let G be a 3-regular graph. If $I_k(G) = \mathbb{Z}_k$ for all $k \geq 3$, then G admits a perfect matching.

Proof.

Assume that G has a $\mathbb{Z}_4$ -magic labelling with index 0. Then, for any vertex, there must be an even number of odd labels incident to that vertex. Thus there must be an odd number (either 1 or 3) of even labels (which must be 2) incident to each vertex.

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Proof.

Assume that G has a $\mathbb{Z}_4$ -magic labelling with index 0. Then, for any vertex, there must be an even number of odd labels incident to that vertex. Thus there must be an odd number (either 1 or 3) of even labels (which must be 2) incident to each vertex. But if all three labels are 2, then the sum at that vertex is 2. So each vertex has exactly one edge labelled 2 incident to it. This is our perfect matching. □

$\mathbb{Z}_k$ -magic labellings of $2r$ -regular graphs

Theorem

Let G be a connected $2m$ -regular graph, with $m > 1$ odd, with an even number of vertices. Then $I_k(G) = \mathbb{Z}_k$ for all $k \geq 3$.

$\mathbb{Z}_k$ -magic labellings of $2r$ -regular graphs

Theorem

Let G be a connected $2m$ -regular graph, with $m > 1$ odd, with an even number of vertices. Then $I_k(G) = \mathbb{Z}_k$ for all $k \geq 3$.

Does not require a perfect matching.

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Theorem

Let G be a connected $2m$ -regular graph, with $m > 1$ odd, with an even number of vertices. Then $I_k(G) = \mathbb{Z}_k$ for all $k \geq 3$.

Does not require a perfect matching.

In the case of all other even regular graphs with an even number of vertices, the only possible holes are 1 and 3 in $I_4(G)$.

Additionally, every graph tested so far has had a full index set $I_4(G)$.

Edge-Antimagic Labellings

Definition

An **edge-antimagic labelling** of a graph is a bijective assignment of the numbers $\{1, \dots, |V|\}$ to the vertices such that the sum of the labels of the endpoints of each edge is different.

What graphs are edge-antimagic?

- ▶ Trees

What graphs are edge-antimagic?

- ▶ Trees
- ▶ $W_n - e$ for $n \leq 6$

What graphs are edge-antimagic?

- ▶ Trees
- ▶ $W_n - e$ for $n \leq 6$
- ▶ K_{1n}, K_{2n}

What graphs aren't edge-antimagic?

- ▶ Graphs with p vertices and more than $2p - 3$ edges

What graphs aren't edge-antimagic?

- ▶ Graphs with p vertices and more than $2p - 3$ edges
- ▶ $W_n - e$ for $n \geq 7$

Deficiency of Graphs

Definition

The **deficiency** of a graph is the minimum number of additional isolated vertices needed to make the graph edge-antimagic.

Deficiencies of Various Graphs

Wheels:

Deficiencies of Various Graphs

Wheels:

$$d(W_3) = 1$$

Deficiencies of Various Graphs

Wheels:

$$d(W_3) = 1$$

$$d(W_4) = 1$$

Deficiencies of Various Graphs

Wheels:

$$d(W_3) = 1$$

$$d(W_4) = 1$$

$$d(W_5) = 1$$

Deficiencies of Various Graphs

Wheels:

$$d(W_3) = 1$$

$$d(W_4) = 1$$

$$d(W_5) = 1$$

$$d(W_6) = 1$$

Deficiencies of Various Graphs

Wheels:

$$d(W_3) = 1$$

$$d(W_4) = 1$$

$$d(W_5) = 1$$

$$d(W_6) = 1$$

$$d(W_7) = 1$$

Deficiencies of Various Graphs

Wheels:

$$d(W_3) = 1$$

$$d(W_4) = 1$$

$$d(W_5) = 1$$

$$d(W_6) = 1$$

$$d(W_7) = 1$$

$$d(W_8) = 2$$

Deficiencies of Various Graphs

Wheels:

$$d(W_3) = 1$$

$$d(W_4) = 1$$

$$d(W_5) = 1$$

$$d(W_6) = 1$$

$$d(W_7) = 1$$

$$d(W_8) = 2$$

$$d(W_9) = 2$$

Deficiencies of Various Graphs

Wheels:

$$d(W_3) = 1$$

$$d(W_4) = 1$$

$$d(W_5) = 1$$

$$d(W_6) = 1$$

$$d(W_7) = 1$$

$$d(W_8) = 2$$

$$d(W_9) = 2$$

$$d(W_{10}) = 2$$

Deficiencies of Various Graphs

Complete graphs:

Deficiencies of Various Graphs

Complete graphs:

$d(K_1) = 0$ (optimal labelling: 1)

Deficiencies of Various Graphs

Complete graphs:

$d(K_1) = 0$ (optimal labelling: 1)

$d(K_2) = 0$ (1,2)

Deficiencies of Various Graphs

Complete graphs:

$$d(K_1) = 0 \text{ (optimal labelling: 1)}$$

$$d(K_2) = 0 \text{ (1,2)}$$

$$d(K_3) = 0 \text{ (1,2,3)}$$

Deficiencies of Various Graphs

Complete graphs:

$$d(K_1) = 0 \text{ (optimal labelling: 1)}$$

$$d(K_2) = 0 \text{ (1,2)}$$

$$d(K_3) = 0 \text{ (1,2,3)}$$

$$d(K_4) = 1 \text{ (1,2,3,5)}$$

Deficiencies of Various Graphs

Complete graphs:

$$d(K_1) = 0 \text{ (optimal labelling: 1)}$$

$$d(K_2) = 0 \text{ (1,2)}$$

$$d(K_3) = 0 \text{ (1,2,3)}$$

$$d(K_4) = 1 \text{ (1,2,3,5)}$$

$$d(K_5) = 3 \text{ (1,2,3,5,8)}$$

Deficiencies of Various Graphs

Complete graphs:

$$d(K_1) = 0 \text{ (optimal labelling: 1)}$$

$$d(K_2) = 0 \text{ (1,2)}$$

$$d(K_3) = 0 \text{ (1,2,3)}$$

$$d(K_4) = 1 \text{ (1,2,3,5)}$$

$$d(K_5) = 3 \text{ (1,2,3,5,8)}$$

$$d(K_6) = 7 \text{ (1,2,3,5,8,13)}$$

Deficiencies of Various Graphs

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$$d(K_1) = 0 \text{ (optimal labelling: 1)}$$

$$d(K_2) = 0 \text{ (1,2)}$$

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$$d(K_4) = 1 \text{ (1,2,3,5)}$$

$$d(K_5) = 3 \text{ (1,2,3,5,8)}$$

$$d(K_6) = 7 \text{ (1,2,3,5,8,13)}$$

$$d(K_7) = 12 \text{ (1,2,3,5,9,14,19)}$$

Deficiencies of Various Graphs

Complete graphs:

$$d(K_1) = 0 \text{ (optimal labelling: 1)}$$

$$d(K_2) = 0 \text{ (1,2)}$$

$$d(K_3) = 0 \text{ (1,2,3)}$$

$$d(K_4) = 1 \text{ (1,2,3,5)}$$

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$$d(K_6) = 7 \text{ (1,2,3,5,8,13)}$$

$$d(K_7) = 12 \text{ (1,2,3,5,9,14,19)}$$

$$d(K_8) = 17 \text{ (1,2,3,5,9,15,20,25)}$$

Deficiencies of Various Graphs

Complete graphs:

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$$d(K_9) > 21$$

Deficiencies of Various Graphs

Complete bipartite graphs:

Deficiencies of Various Graphs

Complete bipartite graphs:

$$\text{If } m, n \text{ odd, then } d(K_{m,n}) \leq \frac{mn}{2} - \frac{m}{2} - \frac{n}{2} + \frac{1}{2}$$

Deficiencies of Various Graphs

Complete bipartite graphs:

$$\text{If } m, n \text{ odd, then } d(K_{m,n}) \leq \frac{mn}{2} - \frac{m}{2} - \frac{n}{2} + \frac{1}{2}$$

$$\text{If } m \text{ is even, then } d(K_{m,n}) \leq \frac{mn}{2} - \frac{m}{2} - n + 1$$

Deficiencies of Various Graphs

Complete bipartite graphs:

$$\text{If } m, n \text{ odd, then } d(K_{m,n}) \leq \frac{mn}{2} - \frac{m}{2} - \frac{n}{2} + \frac{1}{2}$$

$$\text{If } m \text{ is even, then } d(K_{m,n}) \leq \frac{mn}{2} - \frac{m}{2} - n + 1$$

So far, no graphs have been found which have lower deficiencies than these bounds.

References

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