Quadratic Equations over Matrices over the Quaternions

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Fields

- A field consists of a set of objects **S** and two operations on this set.
- We will call these operations addition, denoted by +, and multiplication, denoted by *.
- A field must satisfy the following properties:

Additive Properties

For all **a**, **b**, and **c** in **S**,

- *(Closure)* **a** + **b** is in **S**
- *(Identity)* there exists a “0” in **S** such that **a**+0=**a**=0+**a**.
- *(Inverses)* there exists a “**-a**” such that -**a** + **a** = 0 = **a** + -**a**.
- *(Associative)* **a** + (**b** + **c**) = (**a** + **b**) + **c**
- *(Commutative)* **a** + **b** = **b** + **a**

Multiplicative Properties

For all **a**, **b**, and **c** in **S**,

- *(Closure)* **a** * **b** is in **S**
- *(Identity)* there exists a “1” in **S** such that **a***1=**a**=1***a**.
- *(Inverses)* there exists a “$\mathbf{a}^{-1}$” such that $\mathbf{a}^{-1} * \mathbf{a} = 0 = \mathbf{a} * \mathbf{a}^{-1}$.
- *(Associative)* **a** * (**b** * **c**) = (**a** * **b**) * **c**
- *(Commutative)* **a** * **b** = **b** * **a**

Distributive Property

For all **a**, **b**, and **c** in **S**,

- **a** * (**b** + **c**) = **a*b** + **a*c**
- (**b** + **c**) * **a** = **b*a** + **c*a**

Examples of Field

- Rational Numbers
- Real Numbers
- Complex Numbers

Non-examples of Fields

- Integers: (…,-3,-2,-1,0,1,2,3,…) is not a field -- no multiplicative inverses
- The set of 2 by 2 matrices is not a field.
  - There are sometimes no multiplicative inverses. Consider $\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$
  - Multiplication is not commutative.

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$$

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Division Ring

- A division ring is field that is missing one axiom.
- Multiplication in a division ring does not have to be commutative.
- That is, in a division ring **S**, there can exist **a** and **b** in **S** such that **a*b** does not equal **b*a**.

Example: Quaternions

- The best example of a division ring is the ring of quaternions.
- Call the set of quaternions **H**.
- The quaternions can be thought of as a 4-dimensional vector space over the real numbers with basis {1,i,j,k}
- That is, every quaternion has the form $x=a+bi+cj+dk$ where a,b,c,d are real numbers.

Quaternions: Addition

- Let $x=a+bi+cj+dk$ and $y = e+fi+gj+hk$ be quaternions.
- Then $x+y =(a+e)+(b+f)i+(c+g)j+(d+h)k$.
- Clearly, addition satisfies the field properties.

Quaternions: Multiplication

- Multiplication is more complicated.
- i, j, and k have the properties that:

  $i^2=j^2=k^2=-1$

  $ij=-ji=k$

  $jk=-kj=i$

  $ki=-ik=j$

- Hence, multiplication is not commutative.
- However, the rest of the multiplicative field properties are satisfied.

Quaternions

- The quaternions also satisfy the distributive property.
- Thus the quaternions have all the properties of a field, except multiplication is not commutative.
- Hence, the quaternion are a division ring.

Fundamental Theorem of Algebra

- Consider the polynomial equation

  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 j$

  where a_n, $a_{n-1}, \ldots, a_0$ are complex numbers and a_n does not equal 0.
- By the fundamental theorem of algebra, f(x) has n roots, including multiplicities.

Polynomial Equations over Quaternions

- This is not true over the quaternions.
- A quadratic with quaternion coefficients can have 1, 2, or infinitely many roots.

Polynomial Equations over Quaternions

- Consider $X^2+1=0$, where 0 and 1 are quaternions.
- A subset of solutions is

  $\{ai+bj$: a and b are real, and $a^2+b^2=1\}$
- To see this,

  $(ai+bj)^2+1=(ai+bj)(ai+bj)+1=aiai+baji+abij+bjbj+1=$

  $a^2i^2+baji+abij+b^2j^2=a^2i^2+baji-abji+b^2j^2=$

  $-a^2-b^2+1=-1(a^2+b^2)+1=-1(1)+1=0$
- Thus, there are infinitely many solutions.

Equations over Matrices

- We can also consider polynomial equations over matrices.
- Let $M_2(C)$ be the set of 2 by 2 matrices, with entries that are complex numbers.
- Consider $X^2+AX+B=0$ over $M_2(C)$.
- It is known that the number of solutions is 0,1,2,3,4,5,6, or infinitely many.

Equations over Matrices: Example

Consider the equation: $X^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

A subset of solutions is the set

$\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$: a is real$\}$

Thus, there are infinitely many solutions.

Equation over Matrices: Example

Consider the equation: $X^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

A solution has the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Therefore,

$$X^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Case 1: Suppose **a+d** does not equal 0. Then **b**,**c**=0, so $\mathbf{a}^2=1$ and $\mathbf{d}^2=4$. Therefore, a=1,-1 and b=4,-4.

Case 2. Suppose **a+d**=0. Then $\mathbf{a}^2=\mathbf{d}^2$, so $1=\mathbf{a}^2+\mathbf{bc}=\mathbf{bc}+\mathbf{d}^2=4$. This is a contradiction.

Four Solutions: $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 2 \end{bmatrix}$

Equations over Matrices: Quaternions

- Let $M_2(H)$ be the set of 2 by 2 matrices with entries that are quaternions.
- Now consider $X^2+AX+B=0$ over $M_2(H)$.
- This changes the results from polynomials over $M_2(C)$.
- For example, there are now infinitely many solutions to $X^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

Future Goals

- The goal of my project this is to find out in general how many solutions the polynomial $X^2+AX+B=0$ over $M_2(H)$ has.

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