Polynomial Equations over Matrices over the Quaternions

By: Diana Oliff

Mentor: Professor Robert Wilson

Quaternions

- The quaternions are a division ring – that is, they satisfy all of the axioms of a field except multiplication does not need to be commutative.

Quaternions

- The quaternions can be thought of as a 4-dimensional vector space over the real numbers with basis $\{1,i,j,k\}$
- That is, every quaternion has the form $x=a+bi+cj+dk$ where a,b,c,d are real numbers.
- i, j, and k have the properties that
  - $i^2=j^2=k^2=-1$
  - $ij=-ji=k$
  - $jk=-kj=i$
  - $ki=-ik=j$

Quaternion s

- A quaternion $x=a+bi+cj+dk$ can also be represented by the 2 by 2 complex matrix:

$$\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}$$

Goals

- Let $M_2(H)$ be the set of 2 by 2 matrices over the quaternions.
- My goal was to investigate the number of solutions of quadratic equations over $M_2(H)$

Some Examples

$X^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has no solutions

$X^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ has four solutions: $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 2 \end{bmatrix}$

Some Examples

$X^2+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has infinitely many solutions.

Consider $X=\begin{bmatrix} bi+cj+dk & 0 \\ 0 & b'i+c'j+d'k \end{bmatrix}$ where b,c,d are real.

Then $X^2=\begin{bmatrix} (bi+cj+dk)^2 & 0 \\ 0 & (b'i+c'j+d'k)^2 \end{bmatrix}$

$(bi+cj+dk)^2$

$=b^2i^2+bcij+bdik+cbji+c^2j^2+cdjk+dbki+dckj+d^2k^2$

$=-b^2+bcij-bdki-bcij-c^2+cdjk+bdki-cdkj-d^2$

$=-(b^2+c^2+d^2)$

Some Examples

Thus, $X^2 = \begin{bmatrix} -(b^2+c^2+d^2) & 0 \\ 0 & -(b'^2+c'^2+d'^2) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

There are infinitely many choices for b,c,d,b',c',d' such that this is true.

An Approach

- There exists an injective homomorphism $S: M_2(H) \rightarrow M_4(C)$ given by

$$\begin{bmatrix} a+bi+cj+dk & e+fi+gj+hk \\ m+ni+oj+pk & q+ri+sj+tk \end{bmatrix} \rightarrow \begin{bmatrix} \begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix} & \begin{bmatrix} e+fi & g+hi \\ -g+hi & e-fi \end{bmatrix} \\ \begin{bmatrix} m+ni & o+pi \\ -o+pi & m-ni \end{bmatrix} & \begin{bmatrix} q+ri & s+ti \\ -s+ti & q-ri \end{bmatrix} \end{bmatrix}$$

- Let X' represent the image of X under S.
- If X is a solution to $X^2+AX+B=0$, then X' is a solution to $Y^2+A'Y+B=0$.

The Approach

- Suppose we are given an equation $X^2+AX+B=0$.
- We can find every solution Y to $Y^2+A'Y+B'=0$.
- Some of the solutions Y will not be in the range of S. These do not correspond to solutions of $X^2+AX+B=0$.
- The solutions Y that are in the range of S can be written as an X' for some X that is a solution to $X^2+AX+B=0$.

Solving Quadratics over $M_4(C)$

- How do we find the solutions to $Y^2+A'Y+B'=0$?
- There is a general approach in notes written up by Professor Wilson to finding diagonalizable solutions to quadratics over $M_4(C)$.
- Let R be the set of roots of $\det[t^2+A't+B']=0$.
- $\det[t^2+A't+B']$ is an 8^{th} degree polynomial, and so has 8 roots, counting multiplicities.
- Assume for the remainder of the presentation that the roots are distinct.
- These are the potential eigenvalues of the solution Y to $Y^2+A'Y+B'=0$.

Solving Quadratics over $M_4(C)$

- Let $E=\begin{bmatrix} e_1 & & & 0 \\ & e_2 & & \\ 0 & & e_3 & \\ & & & e_4 \end{bmatrix}$ where e_1, e_2, e_3, e_4 are in R.
- Let $N=\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$ where v_i is an element of the nullspace of $e_i^2 I + A' e_i + B'$.
- If N is invertible, then $Y=NEN^{-1}$ is a solution to $Y^2+A'Y+B'=0$.
- Additionally, every solution of $Y^2+A'Y+B'=0$ has the form $Y=NEN^{-1}$ for some E and corresponding N.

Solving Quadratics over $M_4(C)$

- We assumed that $\det[t^2+A't+B']=0$ has distinct roots.
- Thus the nullspace for each e_i has dimension 1.
- So once E is chosen, so is N.
- There are 70 [8 choose 4] distinct choices for E, and so at most 70 solutions Y for $Y^2+A'Y+B'=0$.
- Hence, there are, at most, 70 solutions to $X^2+AX+B=0$.

Experimental Evidence

- There are two questions that remain:
  - How often is our assumption that the roots of $\det[t^2+A't+B']=0$ are distinct satisfied?
  - How many of the solutions Y to $Y^2+A'Y+B'=0$ are equal to X' for some solution X to $X^2+AX+B=0$?

Experimental Evidence

- To answer these question, we used Mathematica to generate several random matrices A and B in $M_2(H)$.
- We used mathematica to find the roots of $\det[t^2+A't+B']=0$.
- In all of our randomly generated examples, the roots were distinct. Thus, in the random case, our assumption that $\det[t^2+A't+B']=0$ has distinct roots is satisfied.
- We proved that the set of quadratic equations for which the assumption is true is Zariski open.

Experimental Evidence

- We also found that the roots of $\det[t^2+A't+B']=0$ always occur in pairs of complex conjugates.
- We proved that this always happens.
- Given our randomly generated matrices A and B, we used the methods described previously to find the solutions Y to $Y^2+A'Y+B'=0$.
- We then looked at which corresponded to solutions of $X^2+AX+B=0$.
- We found that a solution Y correspond to an X' for some solution X to $X^2+AX+B=0$ iff $Y=NEN^{-1}$, where the diagonal of E consists of two sets of complex conjugates.
- We proved that this must always happen.

Results

- There are exactly 6 (4 choose 2) ways in which this can occur, and therefore, at most six solutions to $X^2+AX+B=0$.
- Thus we have solved the case where $\det[t^2+A't+B']$ has distinct roots.

Further Goals

- We assumed $\det[t^2+A't+B']=0$ has distinct roots. What happens if we lift this requirement? What is the general case?