

Finite Geometries for Those with a Finite Patience for Mathematics

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1 Introduction

1.1 Objective

When my friends ask me what I've been studying this past summer and I tell them that I've been looking at finite geometries, it is disappointing that they all seem to groan at simply the name of the subject. They seem more than willing to tell me about how poorly they did in their high school geometry courses, and that they believe they could never begin understand what I'm studying. The most disappointing part of their response is that most of them will never see some of the beautiful and elementary results that stem from basic finite geometries simply because they assume it is too complicated. Although some of the textbooks on the subject might be difficult for a non math-major to sift through, my intent is to simplify some of the basic results so that any novice can understand them.

1.2 Method

My intent is to compare the axiomatic system of Euclidean geometry that everyone is familiar with to the simpler axiomatic systems of finite geometries. Whereas it is a very intimidating task to show how Euclid's postulates lead to many of the theorems in plane geometry, it is much easier to look at simple cases in finite geometry to show how the axioms shape the geometries. I will prove some simple theorems using these basic axioms to show the reader how axiomatic systems can produce nontrivial theorems, and I will touch on projective planes and affine planes to show examples of more constrained axiomatic systems that produce very elegant results. Through this, the reader will gain an appreciation for axiomatic systems, learn some of the basics of finite geometries, and be introduced to some mathematical concepts that are universally important and useful.

2 Euclidean Geometry

The geometry that practically everybody is familiar with is Euclidean geometry, which in two dimensions is called plane geometry. This is the geometry taught in high school and used by mostly everyone. The basis of plane geometry lies in only five basic facts, called

postulates¹, which govern the laws of how the geometry works and end up creating many complex theorems and beautiful results. These five postulates are as follows:

- Any two points can be joined by a straight line.
- Any straight line segment can be extended indefinitely into a straight line.
- Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
- All right angles are congruent.
- Through a point not on a given straight line, one and only one line can be drawn that never meets the given line².

These five postulates shape all of plane geometry and are the most basic facts that lead to very complex theorems. It is very difficult to understand how these postulates by themselves lead to all of plane geometry because plane geometry is such a broad geometry; however, finite geometries are easier to pick apart and lead to a better understanding of how a geometry is shaped.

3 Introduction to Near-linear Spaces

3.1 Finite Geometries vs. Infinite Geometries

It would make sense to begin by defining finite geometries. **Finite geometry** describes any geometric system that has only a finite number of points³. So of course the plural, finite geometries, is the set of geometric systems that have only finite numbers of points. This is different from plane geometry because plane geometry contains an infinite number of points (x, y) where x and y span the real numbers. Finite geometries are a little more abstract in that they don't cover the entire plane – you just start out with a space. A space $S = (P, L)$ is a system of points P and lines L such that the lines are subsets of the points. For example, consider the following space:

In Figure 1 we have a space with five points labeled 1, 2, 3, 4, and 5. There are six lines. Consider the line connecting points 1 and 2; we label that line $\{1, 2\}$. Similarly, the other lines containing point 1 are labeled $\{1, 3\}$, $\{1, 4\}$, and $\{1, 5\}$. Then there is the line connecting points 2 and 5, giving us $\{2, 5\}$. Lastly there is the line connecting points 3, 4, and 5, which we label $\{3, 4, 5\}$. So we have our space $S = (P, L)$ with $P = \{1, 2, 3, 4, 5\}$ and $L = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 5\}, \{3, 4, 5\}\}$. Since L is a set of lines, and each line is a set of points, L must be written as a composition of sets. Thus we have Figure 1 as our first example of a finite geometry.

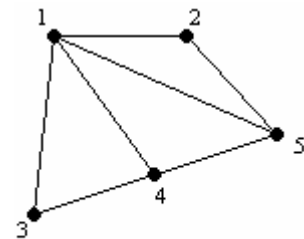


Figure 1

¹ Eric W. Weisstein. "Euclid's Postulates." From *Mathworld*—A Wolfram Web Resource. <http://mathworld.wolfram.com/EuclidsPostulates.html>

² Although this is not Euclid's original fifth postulate, it is an equivalent statement to his fifth postulate and more easily understandable.

³ "Finite Geometry." From *Wikipedia*. http://en.wikipedia.org/wiki/Finite_geometry

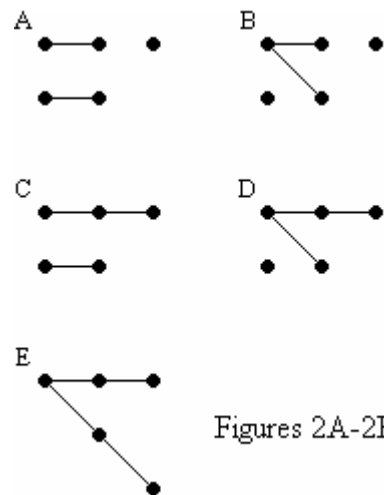
It is very important to note here that $\{3, 4, 5\}$ is one line containing three points. If we were dealing with two different lines containing only two points each, we would label them $\{3, 4\}$ and $\{4, 5\}$ and draw them so that they didn't look like one line. The diagrams contained henceforth are carefully drawn to distinguish between these two cases.

3.2 Axiomatic Systems

In the same way plane geometry is based on postulates, finite geometries are based on certain given facts called axioms. Although we could list some of the axioms that the space represented by Figure 1 satisfies, the list would continue forever. We could include axioms like: each line must contain no more than three points; there exist five points; and there exists no line containing twenty seven points. It is much more practical and interesting to start with a set of axioms and see which spaces satisfy them. Consider the following axiomatic system:

1. There are five points and two lines.
2. Each line contains at least two points.
3. Each line contains at most three points.

It might seem that there would be many possible spaces that follow the axioms, but we can systematically consider the possibilities. We know that each line must contain at least two points. So we first consider all spaces in which both lines contain two points each, discovering the spaces represented by Figures 2A and 2B. Looking at spaces in which one line contains two points and the other line contains three points gives us Figures 2C and 2D. And looking at spaces in which both lines contain three points gives us Figure 2E. Since each line contains at most three points, we know we have considered all possible spaces.



Figures 2A-2E

3.3 Standard Notation

Mathematicians working with finite geometries have a few standard ways to represent spaces. This is more technical than mathematically significant, but it helps to keep things in order. For example, when we looked for all spaces that satisfied the previous axiomatic system, we looked at them by the number of points on each line, in increasing order. This is one way to ensure uniqueness of each space, and to guarantee that we have found each space.

Another practice we have already seen comes into play when representing a space in set notation. When we described Figure 1 as S , we wrote out the points in numerical order: $P = \{1, 2, 3, 4, 5\}$. But when we labeled lines, we listed the points within each line in

numerical order: $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 5\}$, and $\{3, 4, 5\}$. We also listed the lines themselves in a numerical order, starting with all lines that contain point 1, then all other lines that contain point 2, and so on. These are just some standard ways for working with finite geometries.

3.4 Consistency and Dependence

Additionally, we must touch on consistency and independence. Consistency is whether or not a set of axioms produces a space. A **consistent** system, like the one we just saw, produces at least one space. An **inconsistent** system does not. One example of an inconsistent system is the following:

1. There exist four points and six lines.
2. There is at least one line with three points.

When looking at all spaces with four points, we see that the only space that contains four points and six lines is the space $S = \{P, L\}$ where $P = \{1, 2, 3, 4\}$ and $L = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. Because no lines have three points, no such system satisfies this axiomatic system, and this system is inconsistent. For our purposes, we will only deal with systems that are consistent.

Additionally we can discuss the independence of axiomatic systems. **Independent** systems are systems in which no postulate can be derived from the others. Every system we have dealt with so far has been independent. A **dependent** system is one in which at least one of the axioms can be derived from the other axioms. For example, the following axiomatic system is dependent:

1. There are five points and three lines.
2. Each line has at most five points.

This system is dependent because the second axiom follows directly from the first axiom. As long as a space has only five points, no line can contain more than five points. In this case, it is trivial to see that the system is dependent. In other cases, a good way to test dependency is to consider the systems formed using all but one of the axioms. If a subset of the axioms produces the same spaces that the entire set of axioms produces, then the set is dependent.

Consistency and dependence are not too important when studying basic finite geometries because it only makes sense to work with consistent, independent systems. So although it is important to be familiar with those terms, intricately testing the independence of an axiomatic set is not of utter importance.

3.5 Near-linear Spaces

The broadest axiomatic system of finite geometries that mathematicians deal with is the set of near-linear spaces. A **near-linear space** is a space of points and lines such that any

line contains at least two points and two points are on at most one line. These axioms might seem trivial, but one can build numerous theorems upon them. Also, they guarantee uniqueness between a space and its geometrical representation.

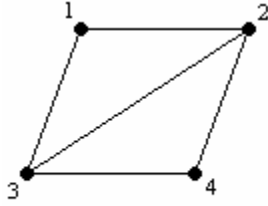


Figure 3

The first axiom, which states that any line contains at least two points, eliminates the possibility of having a line with only one point. We can see that Figure 2 represents the space $S = \{P, L\}$ such that $P = \{1, 2, 3, 4\}$ and $L = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. This axiom eliminates the possibility that L contains $\{1\}$, $\{2\}$, $\{3\}$, or $\{4\}$ and guarantees that the space represented by Figure 2 is unique.

The second axiom, stating that two points are on at most one line, eliminates the possibility of having two lines between two points. If we were to look at the space $S = \{P, L\}$ for which $P = \{1, 2, 3\}$ and $L = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, we can see that it would be a simple triangle. However, Figure 3 contains all and only all of the lines as the space we considered, but it is certainly not what was intended by S . This axiom makes sure that when someone is given a space $S = \{P, L\}$ it has a unique geometrical representation.

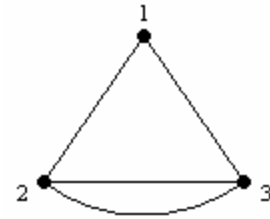


Figure 4

These axioms, which we shall refer to as NL1 and NL2 respectively, are the basis for near-linear spaces and the foundation for the following principles and ideas.

4 Properties of Near-linear Spaces

4.1 Basic Lemmas

Using only the two axioms for near-linear spaces, we can start constructing some lemmas about near-linear spaces:

Lemma 1: Two distinct lines of a near-linear space intersect in at most one point⁴.

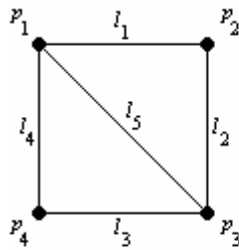
Proof: Consider two lines in a near-linear space, l_1 and l_2 . Suppose l_1 and l_2 intersect at two points, call them p and q . But this would mean that both p and q are l_1 and l_2 , meaning that these two points are on two lines. This contradicts NL2, which states that two points are on at most one line. Therefore our supposition was incorrect, so l_1 and l_2 must not intersect at two points.

Lemma 2: If the points on l_1 are a subset of the points on l_2 , then $l_1 = l_2$.

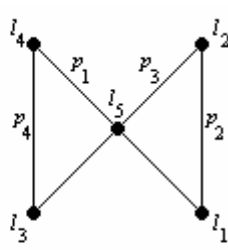
Proof: We know from NL1 that each line must contain at least 2 points. So we know that if l_2 contains the points on l_1 , then l_2 and l_1 must have at least two points in common. But we already saw that two lines cannot have two points in common because that would contradict NL2. Therefore we know that l_1 and l_2 must be one line, so $l_1 = l_2$.

4.2 The Dual Space

One really interesting property of finite geometries is that there is a way to interchange points and lines from one space to create a new space. For example, we have Figure 5A with four points, labeled p_1 through p_4 , and five lines, labeled l_1 through l_5 . If we were to interchange points with lines, we would have a new figure with five points and four lines. It is a little tricky to write this geometrically, but it is easy to see that Figure 5B would be equivalent to this new space.



A



B

Figure 5

In Figure 5A, we can see that p_1 touches l_1 , l_4 , and l_5 , p_2 touches l_1 and l_2 , p_3 touches l_2 , l_3 , and l_5 , and p_4 touches l_3 and l_4 . In Figure 5B, we can see that p_1 contains l_1 , l_4 , and l_5 , p_2 contains l_2 and l_5 , p_3 contains l_2 , l_3 , and l_5 , and p_4 contains l_3 and l_4 . Since this is true, we know that we have correctly interchanged the points and lines in the spaces. And since the space represented by Figure 5B satisfies NL1 and NL2, it is a near-linear space.

Of course, we cannot always simply interchange points and lines and end up with a space. For example, if our space were simply two points connected by a line, interchanging points and lines would give us only one point with two lines jetting out of it into nowhere, and that does not satisfy NL1. We can take care of this by first transforming the lines of our given space into points in our new space. Then transform all points of our given space that fall on at least two lines, and only those points, into lines in our new space. This new space is called the **dual space** of our original near-linear space, and it is simple to prove that the dual space of a near-linear space is also a near-linear space.

Lemma 3: The dual space of a near-linear space is a near-linear space.

Proof: NL2 tells us that in our original space, two points are on at most one line. Since we are simply interchanging points and lines to create the dual space, we know that two lines intersect in at most one point in the dual space. Since two lines cannot intersect at two or more points, we know

that we cannot have two points contained on two lines. Thus two points are on at most one line, and NL2 is satisfied⁵.

Before we transformed the original space into the dual space, we removed any point that touched only one line. This means that after the transformation, all lines containing only one point were removed. Thus all lines contain at least two points, and NL1 is satisfied.

So we have already seen how a simple axiomatic system can lead nontrivial theorems. Although Lemmas 1 and 2 may have been intuitive, Lemma 3 produced some more complicated results.

4.3 Subspaces

We just saw how to create a unique near-linear space from one we already had, but there are other special near-linear spaces we can create called subspaces. A **subspace** of a space $S = (P, L)$ is a subset X of points of P such that for any two points X that are on the same line, all of the points on that line must be in X . Some trivial subspaces are the empty set, any point in the space, any single line in the space, and the entire space itself.

Let's look for the subspaces in Figure 6. The trivial subspaces are $\{\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{1, 2, 4\}$, $\{1, 3, 7\}$, $\{2, 3\}$, $\{2, 5\}$, $\{2, 6\}$, $\{3, 5\}$, $\{3, 6\}$, $\{4, 5, 6, 7\}$, and $\{1, 2, 3, 4, 5, 6, 7\}$. If we try to form a new subspace, we can use at most one point from each of the exterior lines $\{1, 2, 4\}$, $\{1, 3, 7\}$, and $\{4, 5, 6, 7\}$ because including two points from any of those lines would force our subspace to be either one of those lines or the entire space. For example, if we included 5 and 6, we'd be forced to include the line $\{4, 5, 6, 7\}$, and any other point would force us to include the entire space. Therefore the only other subspaces are $\{1, 5\}$, $\{1, 6\}$, $\{2, 3, 5\}$, $\{2, 3, 6\}$, $\{2, 7\}$, and $\{3, 4\}$.

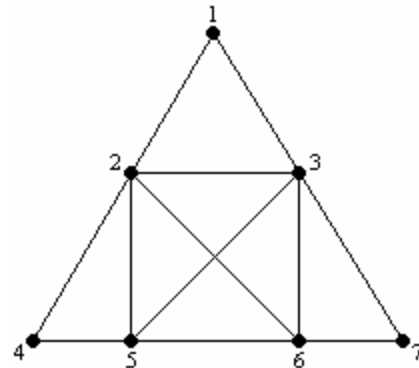


Figure 6

Listing subspaces can be very tricky, as even the slightest change can affect everything. Consider what happens to the subspace $\{2, 3, 5\}$ we had before when we add a point where the lines $\{2, 6\}$ and $\{3, 5\}$ overlap. Label this point 8. Since we already included two points on line $\{3, 5, 8\}$, we must include the entire line. We would then have to include point 6 because then we would be including two points on line $\{2, 6, 8\}$. At that point our subspace would include two points on line $\{4, 5, 6, 7\}$, forcing us to include the entire line and finally the entire space. We have just used a subset of the points in a space to generate a new subspace, and this subspace is called the closure of $\{2, 3, 5\}$.

⁵ In fact, the statements “two points are on at most one line” and “two lines intersect in at most one point” are equivalent. In the proofs of Lemmas 1 and 3 we have shown how each statement implies the other. For a geometrical representation of this, examine the validity of these statements in regards to Figure 4.

4.4 Closure

The **closure**⁶ of a subset of points X is the smallest subspace containing X , and it is denoted $\langle X \rangle$. As we just saw, we can take any set X and generate its closure. We do this by looking at the points contained in X , and for any set of collinear points, we include all other points on that line. There are some trivial closures, as there were some trivial subspaces. The closure of any point is itself, and the closure of any set of two or more collinear points is the line containing those points. But when we look at other types of sets of points, we begin to actually generate their closures.

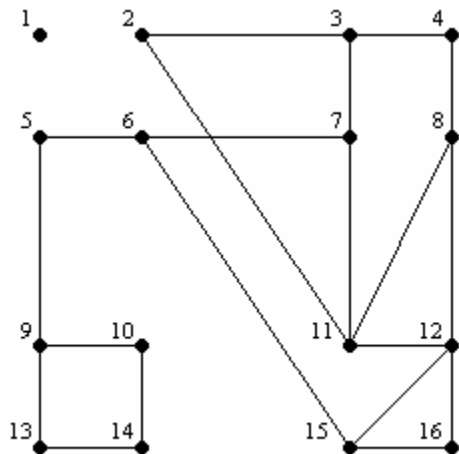


Figure 7

Consider the near-linear space in Figure 7. Let us generate the closure for $X = \{2, 3, 11, 16\}$. Since we already have $\{2, 3\}$, we must include $\{2, 3, 4\}$. Then we can see that $\{3, 11\}$ gives us $\{3, 7, 11\}$, and $\{4, 16\}$ gives us $\{4, 8, 12, 16\}$. Therefore $\langle \{2, 3, 11, 16\} \rangle = \{2, 3, 4, 7, 8, 11, 12, 16\}$. Additionally, we can easily see that $\langle \{3, 6, 12, 15\} \rangle = \{3, 6, 12, 15\}$. But if we add points 9 and 13 and look at $X = \{3, 6, 9, 12, 13, 15\}$, we see that $\{9, 13\}$ gives us $\{5, 9, 13\}$, $\{5, 6\}$ gives us $\{5, 6, 7\}$, and $\{3, 7\}$ gives us $\{3, 7, 11\}$, so $\langle \{3, 6, 9, 12, 13, 15\} \rangle = \{3, 5, 6, 7, 9, 11, 12, 13, 15\}$. And if we were to add point 2, 4, 8, or 16 to X , our closure would include all points except 1, 10, and 14.

One good consequence of dealing with closures is that the closure of any set of points is a subspace of the near-linear space. Therefore we can learn more about how subspaces work through examining the process of generating closures. But also, generating closures helps us define the concept of dimension in near-linear spaces, which is very abstract compared to the intuitiveness of dimension in Euclidean geometry.

4.5 Dimension

We know that anything in plane geometry can have at most two dimensions. The reasoning most people might give for this is that each point in the plane can be represented by a pair of coordinates (x, y) , for all real numbers x and y . Because there are two variables, we know there are two dimensions. But what about finite geometries? So far we have represented the points we have used by assigning them one whole number $\{x\}$ as a label. Since we have used only one variable, are near-linear spaces one-dimensional. Or are they two-dimensional because we have drawn them all in the plane? It looks like we need to take a closer look into what makes plane geometry two-dimensional.

⁶ Although the proper definition of a closure of X is a subspace which contains X but does not properly contain any subspace on X , the definition given above is more intuitive for our purposes.

We just said that each point in the plane can be represented by a pair of coordinates. If we look at the plane and consider two vectors⁷, $\mathbf{e}_1$ and $\mathbf{e}_2$ as pictured in Figure 8, we can see that any point (x, y) is equivalent to $x\mathbf{e}_1 + y\mathbf{e}_2$ for all real-valued scalars x and y . For example, the point $(2, 3)$ is constructed by adding the twice the vector $\mathbf{e}_1$ to three times the vector $\mathbf{e}_2$, giving us $2\mathbf{e}_1 + 3\mathbf{e}_2$. Similarly, any point in plane geometry can be formed by adding scalar multiples of these two vectors. We define this space to be two-dimensional because we need *at least* two vectors to form the entire space, and we say that those two vectors are the **basis** for the space.

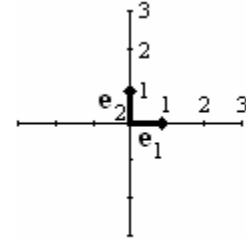


Figure 8

There is a corresponding method for defining dimension in near-linear spaces. Define a **basis** of a near-linear space to be a set of points that generates the space. A basis is not necessarily unique, nor do all bases of a space contain the same number of points. For example, some of the bases for the space in Figure 6 are $\{1, 2, 3\}$, $\{1, 4, 5\}$, and $\{2, 3, 5, 6\}$. We can see how the set of points generating the space is similar to vectors generating spaces in Euclidean geometry. Looking at the basis $X = \{1, 2, 3\}$, we can see how to generate the space with our origin being $\{1\}$, and with our two vectors $\{1, 2\}$ and $\{1, 3\}$. We begin with $\{1, 2, 3\}$ and we accumulate 4 and 7 from the vectors $\{1, 2\}$ and $\{1, 3\}$, respectively. Then we accumulate 5 and 6 from a combination of the two vectors. So we only needed the two vectors to generate the entire space. Therefore, the space is two-dimensional.

We can now define the **dimension** of a near-linear space to be one less than the number of points in its smallest basis. We saw that the first point acts as the origin, and each additional point acts as a vector. Therefore if our basis contains n points, it contains $n - 1$ vectors and our space is $(n - 1)$ -dimensional. So we can see that the space in Figure 1 is three-dimensional and the space in Figure 7 is nine-dimensional.

4.5 Conclusion of Near-linear Spaces

We have just taken a look at the broadest category of finite geometries that is commonly dealt with. We have seen how an axiomatic system can lead to very interesting properties, and we have seen how the results we obtained are relevant to other types of geometry and other areas of mathematics. Next we will look at two of the more specific and commonly studied types of near-linear spaces, whose axiomatic systems produce strikingly unique results.

5 Affine Planes

Although near-linear spaces have provided us with many interesting properties to consider given simply some points and lines, there are more restricted axiomatic systems

⁷ A vector is a quantity with both a magnitude and a direction. Scalars are simply quantities without direction. So in this example, the two vectors signify moving one unit horizontally or vertically, and the scalars that these vectors are multiplied by indicate how far in each direction we move.

that can give us great results. One of these axiomatic systems produces affine planes. An affine plane is a geometry based on the following axiomatic system:

- A1. Any line contains at least two points.
- A2. Two points are on precisely one line
- A3. Any point p not on a line l is on one line not intersecting l .⁸
- A4. There exists a set of three non-collinear points.

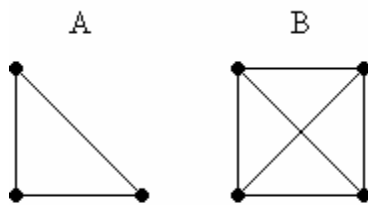


Figure 9

So instead of just showing an example of an affine plane, let's construct one. We start with the three non-collinear points from A4, and A2 tells us to connect the dots, which gives us the space in Figure 9A. Then A3 tells us that we must have lines parallel our three, so we need a fourth point. Once we add this fourth point, we can see how to add the three lines parallel to the ones we already have. We now have the space in Figure 9B, which is an affine plane. It is easy to see that the axioms are satisfied.

But where can we go from here? Can we have any more affine planes with two points on each line? We cannot add any more lines to the points already in our space because we would violate A2. Therefore, we must add a point to our space, giving us the new space in Figure 10A. Since we know that every two points must be on precisely one line, we must draw additional lines between our new point and each of our old points. But focus on the two additional lines drawn in Figure 10B. These two lines are on the additional point we drew, and they are both parallel to the bottom horizontal line in our space. This is a violation of A3. Therefore, there exist no more affine planes with two points on each line.

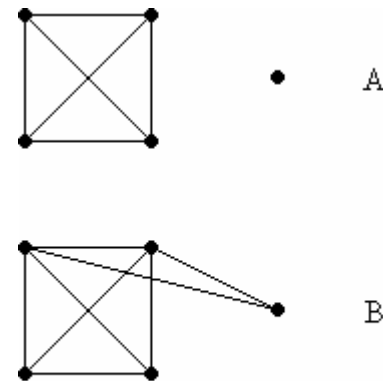


Figure 10

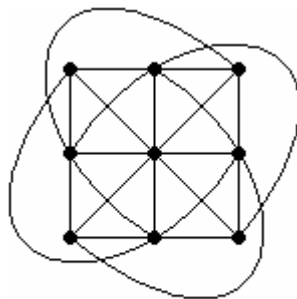


Figure 11

The space we constructed in Figure 9B is known as the affine plane of order 2. There are four points and six lines. The lines are such that there are three sets of two parallel lines, so we say that there are three **parallel classes** with two lines in each class. The affine plane of order 3 is shown in Figure 11, and as you can see it has nine points and twelve lines, in four parallel classes of three lines each. In general, the affine plane of order n has n^2 points and $n^2 + n$ lines in $n + 1$ parallel classes of n lines each. The idea of parallel classes is important to understand how to construct affine planes of order p , for all primes p .

⁸ This theorem guarantees the existence of parallel lines as they are in plane geometry: for each point not on line, there is one line on that point that is parallel to the original line.

We want to find an affine plane of order p , so we know we must have $p + 1$ parallel classes. Figure 12 shows one line of each of those parallel classes for the affine plane of order 5. The first line simply moves over. The second line moves over one point and up one point. The third line moves over one point and up two points. And this continues until we get to the vertical line. Notice that each line contains one point from each row and each column, except for the horizontal and vertical parallel classes. The parallel classes are formed by creating new lines that are shaped in the same way that these lines are, but moving them over horizontally (except for the horizontal line, which is moved vertically).

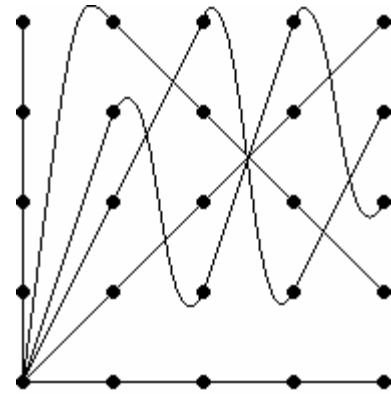


Figure 12

Although it is hard to show for orders higher than three, it is easy to see for any prime order p because the construction we used works for all primes. The reason this construction does not work for composites is that we would have lines with more than one point in each row.

6 Projective Planes

Another very interesting axiomatic system produces projective planes. The axioms are as follows:

- P1. Any line contains at least two points.
- P2. Two points are on precisely one line.
- P3. Two lines intersect at precisely one point.
- P4. There exist a set of four points, no three of which are collinear.

As we can see, many of the axioms for projective planes are the same as those for affine planes. We can see that A1 $\sim$ P1 and A2 $\sim$ P2, but also note that A4 and P4 both simply guarantee that the planes are nonempty. So the main constructive difference is that affine planes have parallel lines and projective planes do not have any. The easiest way to turn an affine plane into a projective plane is to add one point for each set of parallel classes, extend each line in the parallel classes to that point, and form one new line consisting of all of the new points. We shall construct the projective plane of order two from the affine plane of order two.

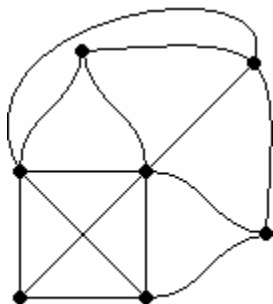


Figure 13

Recall that the affine plane of order two is four points with each pair of points connected. We can see this affine plane as a subspace of the space in Figure 14. The parallel class of horizontal lines has been extended to meet at a point, on the right side of the figure. Similarly, the vertical parallel class intersects at the top of the figure, and the diagonal parallel class intersects on the upper-right side of the figure. Additionally, since these three newly created points are not collinear, we make them form a line.

Although the picture isn't too pretty, it is easy to see that all of the axioms are satisfied in this figure. A more elegant representation of the projective plane of order two can be seen in Figure 14, more commonly known as the "Fano Plane." Also, we can see that there are seven points, each of which is on three lines, and there are seven lines, each of which contains three points.

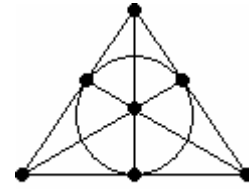


Figure 14

In general, we note that the projective plane of order p contains $p^2 + p + 1$ points and $p^2 + p + 1$ lines. Each point is on $p + 1$ lines, and each line contains $p + 1$ points. It has been shown that a projective plane of order p exists when and only when an affine plane of order p exists. It is conjectured that these planes exist and only exist for orders p^n for prime numbers p and natural numbers n , but this has yet to be proven. It is said that this is one of the most important problems in combinatorics; however, it gets very difficult to show that there exists no affine or projective plane for a certain order, even for small orders. As of now, researchers have shown that no plane of order ten exists, but it has yet to be shown that no plane of order twelve exists.

7 My Summer Undergraduate Research Experience

Although it has been somewhat of a turbulent summer, I am very happy with how the experience has been. I have learned most of the basics of finite geometry, I have taken a look at an unsolved problem that has really sparked my interest, and I have stumbled upon a few interesting possibilities with what I'd like to do with my life. Although I may not have produced any new significant results, the experience has certainly enhanced my view of mathematics, and I feel that as a result I have matured.

When I first looked into the Summer Undergraduate Research Experience, I thought that it would be a great experience to teach myself an interesting area of mathematics, and to dabble in some unexplored areas. Unfortunately, things became difficult very quickly. I had been tutoring for a few months, and on top of that I took a short-term job writing standardized test questions, figuring that all of them would be good opportunities for me to participate in. Things were going well, and I was coming along well learning the basics of finite geometries; however, the workload became overwhelming when I became ill in July. I was having a very difficult time, and it was a very difficult couple of months.

After I found out I needed to write a report about what I did over the summer, it did finally hit me that I covered a lot of material. It has been so difficult to fit in everything that I have learned (there were a few areas that I did not include, simply because I had no time, and I ran short of time writing the last two sections). Writing about everything that I've learned has made me realize that I have learned a lot this summer, and showing this paper to some of my friends to look at has made me very happy because some of them would have never normally looked into finite geometries, and now they got a brief taste of it.

It has also given me a few ideas about what I might want to do after college. I had been thinking about becoming a professor, but I was never sure I would want to teach some of the math classes I have taken. I really do think that I would enjoy teaching a course on finite geometries. Also, writing this has made me consider possibly writing textbooks. It seems like so many students are scared away from learning math early on, and there aren't enough math books out there written for non math majors. Not that I know if I'd be good at it, or that I'd enjoy it, but now I have the idea in my head that I could become a professor and write textbooks.

Although this project has not gone the way I had intended, I am happy with the outcome. I have learned a lot about finite geometries, some of my friends have been introduced to the subject, and I have learned more about what I like about mathematics. And now that I do have a good basis of knowledge in finite geometries, I am planning on exploring the subject further and seeing where it leads me.