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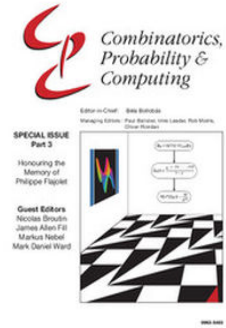

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On Triangle Contact Graphs†

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For Paul Erdős on his 80th birthday

It is proved that any plane graph may be represented by a triangle contact system, that is a collection of triangular disks which are disjoint except at contact points, each contact point being a node of exactly one triangle. Representations using contacts of T- or Y-shaped objects follow. Moreover, there is a one-to-one mapping between all the triangular contact representations of a maximal plane graph and all its partitions into three Schnyder trees.

1. Introduction: on graph drawing

An old problem of geometry consists of representing a simple plane graph G by means of a collection of disks in one-to-one correspondence with the vertices of G. These disks may only intersect pairwise in at most one point, the corresponding contacts representing the edges of G. The case of disks with no prescribed shape is solved by merely drawing for each vertex v a closed curve around v and cutting the edges half way. The difficulty arises when the disks have to be of a specified shape. The famous case of circular disks, solved by the Andreev–Thurston circle packing theorem [1], involves questions of numerical analysis: the coordinates of the centers and radii are not rational, and are computed by means of convergent series. This problem is still up to date, and considered in many research works. In the present paper we will consider **triangular** disks. A **contact point** (A, B) is a node of the triangle A and belongs to the side of the triangle B (but is not a node of B). The asymmetry of the pair (A, B) defines an orientation of the corresponding edge. Such an arrangement is called a **triangle contact system** (see Figure 1). It is obvious that any triangle contact system S defines an oriented simple plane graph $G(S)$. Our result is that any simple plane graph may be represented by a triangle contact system, and that these representations for a maximal plane graph are in one-to-one correspondence with the Schnyder partitions (see definition below).

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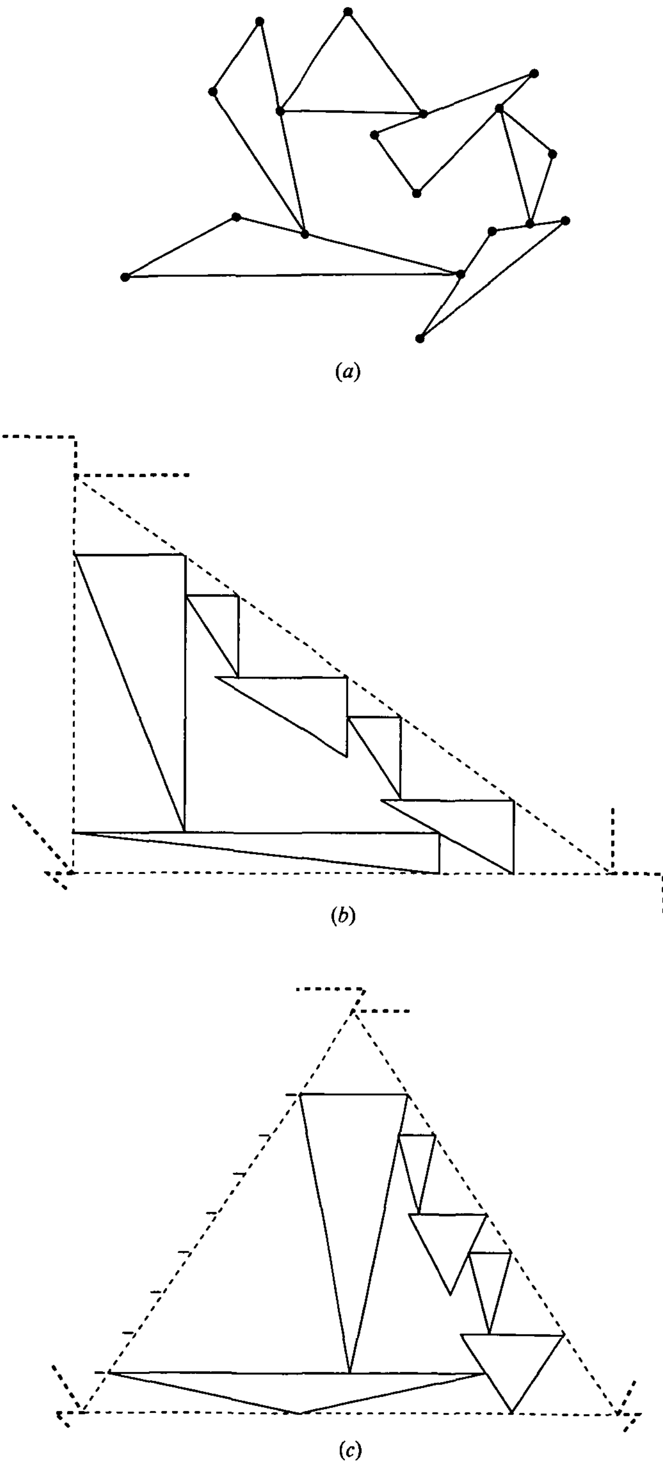


Figure 1 (*a*) A triangle contact system (*b*) A common angle representation of the system 1a (*c*) An isosceles representation of the system 1a

The main tools we shall make use of are the so-called canonical or shelling order of the vertices of a maximal plane graph G introduced in [6], and a partition of the interior edges of G into three trees, due to Schnyder [9].

2. Triangle contact systems and Schnyder partitions

We consider planar objects defined as closed 2-cells. The instances of planar objects appearing below are closed triangles, segments. We also consider T-shaped objects, or Y-shaped objects as limit cases.

Definition. A **contact system** is a finite family of planar objects such that two objects of the family intersect in at most one point, and that three objects have no common point.

A consideration of the tubular neighborhoods of the objects shows that a contact system S defines a unique simple plane graph $G(S)$. Each bounded face of $G(S)$ corresponds in S to a bounded hole of the representation, the unbounded face of $G(S)$ corresponds in S to the unbounded hole. The system S is **biconnected** if $G(S)$ is biconnected.

Definition. A **triangle contact system** S is a contact system such that every object is a closed triangle, and such that each contact point is a node of exactly one triangle. A **subsystem** of S is a family of triangles of S. A **free node** is a node of a triangle which is not a contact point.

A triangle contact system S is **maximal** if and only if each hole of S is delimited by exactly three triangle sides.

Notice that the graph $G(S)$ defined by a maximal triangle contact system S is a maximal planar graph†.

Two triangle contact systems S and S' are **isomorphic** if there exists an isomorphism mapping of the sides of the triangles of S into the sides of the triangles of S'.

Definition. A **canonical order**, or **shelling order**, of the vertices of a maximal plane graph G with external face u, v, w is a labelling of the vertices $v_1 = u, v_2 = v, v_3, ..., v_n = w$ meeting the following requirements for every $4 \leq k \leq n$:

- the subgraph $G_{k-1} \subset G$ induced by $v_1, v_2, \ldots, v_{k-1}$ is 2-connected, and the boundary of its exterior face is a cycle C_{k-1} containing the edge uv;
- the vertex v_k belongs to the exterior face of G_k, and its neighbors in G_{k-1} form a subinterval of the path $C_{k-1} - uv$, with at least two-elements.

It is proved in [6] that such a labelling is always possible and can be computed in linear time by packing the vertices one by one.

† The converse is not true, as pointed out by the referee: K_3 may be represented by a non-maximal triangle contact system

Given a biconnected triangle contact system S and a bounded hole H of S, let t_H be the number of triangles adjacent to H and let p_H be the number of free nodes belonging to the boundary of H.

Theorem 2.1. *A biconnected triangle contact system S is isomorphic to a subsystem of a maximal triangle contact system if and only if*

$$t_H - p_H = 3 \tag{1}$$

for any bounded hole H of S.

Actually, one can prove by simple counting that, if (1) holds for any bounded hole, we have on the unbounded hole H_∞:

$$t_{H_\infty} - p_{H_\infty} = -3 \tag{2}$$

In order to prove necessity, we need the following lemma.

Lemma 2.1. *Let G be a maximal planar graph and G' be a biconnected subgraph of G. Then there exists a sequence of k biconnected subgraphs of G: $G_1 = G, \ldots, G_i, \ldots, G_k = G'$ obtained at each step by deleting exactly one vertex.*

Proof. This lemma is a straightforward consequence of the shelling packing order applied to G while starting from G'. □

This lemma implies that a subsystem S' of a maximal triangle contact system S can be obtained by deleting triangles T_i one by one, keeping the subsystems S_i biconnected.

Proof of theorem 2.1. The proof of necessity is by induction on the number of removed triangles. The property holds for a maximal triangle contact system, since, for a hole H, $t_H = 3$ and $p_H = 0$. Assume the property holds when the first $i-1$ triangles are removed, and consider the removal of the i^{th} triangle T_i from S_{i-1} and the corresponding hole H. As S_{i-1} is biconnected, each triangle adjacent to T_i in S_{i-1} is adjacent to exactly two holes adjacent to T_i. Therefore, the number d of holes adjacent to T_i equals the number of triangles adjacent to T_i, that is the degree of T_i. Let $H_1, \ldots, H_j, \ldots, H_d$ denote these holes. We have

$$\begin{aligned} t_H - p_H &= \left(\sum_{j=1}^{d} t_{H_j} - 2d\right) - \left(\sum_{j=1}^{d} p_{H_j} + d - 3\right) \\ &= \sum_{j=1}^{d} (t_{H_j} - p_{H_j}) - 3d + 3, \end{aligned}$$

that is $t_H - p_H = 3$. Thus, (1) holds for H.

The condition is sufficient. Let S be a biconnected triangle contact system satisfying condition (1), H a bounded hole of S, and let t_H^i denote the number of triangles adjacent to H and having i free nodes in the boundary of H. We shall prove that condition (1) allows us to fill up H with triangles. There exists a triangle T_0 adjacent to H and without free nodes in H. If the hole H is not yet filled up, T_0 can be chosen in such a way that it is adjacent to a triangle T_1 having at least one free node in H. We consider three cases:

— T_1 has three free nodes in H. Add a triangle T in contact with T_0 and T_1, in the position displayed in Figure 2a; the number t_H^3 decreases.

— T_1 has two free nodes in H. Add a triangle T in contact with T_0 and T_1 in the position displayed in Figure 2b; t_H^2 decreases and t_H^3 remains unchanged.

— T_1 has one free node in H. Add a curve-sided triangle T in contact with T_0 and T_1 in the position displayed in Figure 2c; t_H decreases, while t_H^2 and t_H^3 remain unchanged.

At each step the contact system remains biconnected, and as $t_H^2 + t_H^3 + t_H$ decreases, the iteration stops with $t_H = 3$. Equation (2) allows us to fill up the unbounded hole H_∞ by a similar process. The final system obtained can be redrawn as a maximal triangle system (see Section 3), as required. □

One more definition given in [9]:

Definition. A **Schnyder realizer** of a maximal plane graph is a partition of the interior edges of G in three sets Y_r, Y_g, Y_b of directed edges such that for each interior vertex v

— v has indegree one in each of Y_r, Y_g, Y_b,

— the counterclockwise order of the edges incident on v is: entering in T_r, leaving in T_b, entering in T_g, leaving in T_r, entering in T_b, leaving in T_g.

The first condition of the definition implies that Y_r, Y_g and Y_b are three trees oriented from their roots. Schnyder proved in [9] that any maximal plane graph has a realizer.

Any Schnyder realizer may be extended into a **Schnyder partition** by assigning the edges of the exterior face to the three trees in such a way that all the edges of the graph are partitioned into three trees.

In the following, O_r, O_g and O_b will denote the partial orders on $V(G)$ induced by the three oriented trees of a Schnyder realizer, and $\overline{O_r}, \overline{O_g}$ and $\overline{O_b}$ will denote the reversed partial orders.

Now we relate triangle contact systems and Schnyder partitions.

Theorem 2.2. *A maximal triangle contact system S defines a Schnyder partition of $G(S)$.*

For the proof of this theorem we need a lemma. Given a maximal triangle contact system S, let v be a non-free node of a triangle T, belonging to the side of a triangle T', and let f be the mapping that associates with v the node of T' opposite to v (see Figure 3).

Lemma 2.2. *The mapping f is acyclic.*

Proof. We prove a stronger result: there exists no elementary cycle of nodes $N_1, \ldots, N_k$ such that $N_{i+1} = f(N_i)$ and such that N_1 and N_k belong to a common triangle. Such a cycle of k nodes, if it exists, defines a cycle of k triangles. The deletion of the triangles of

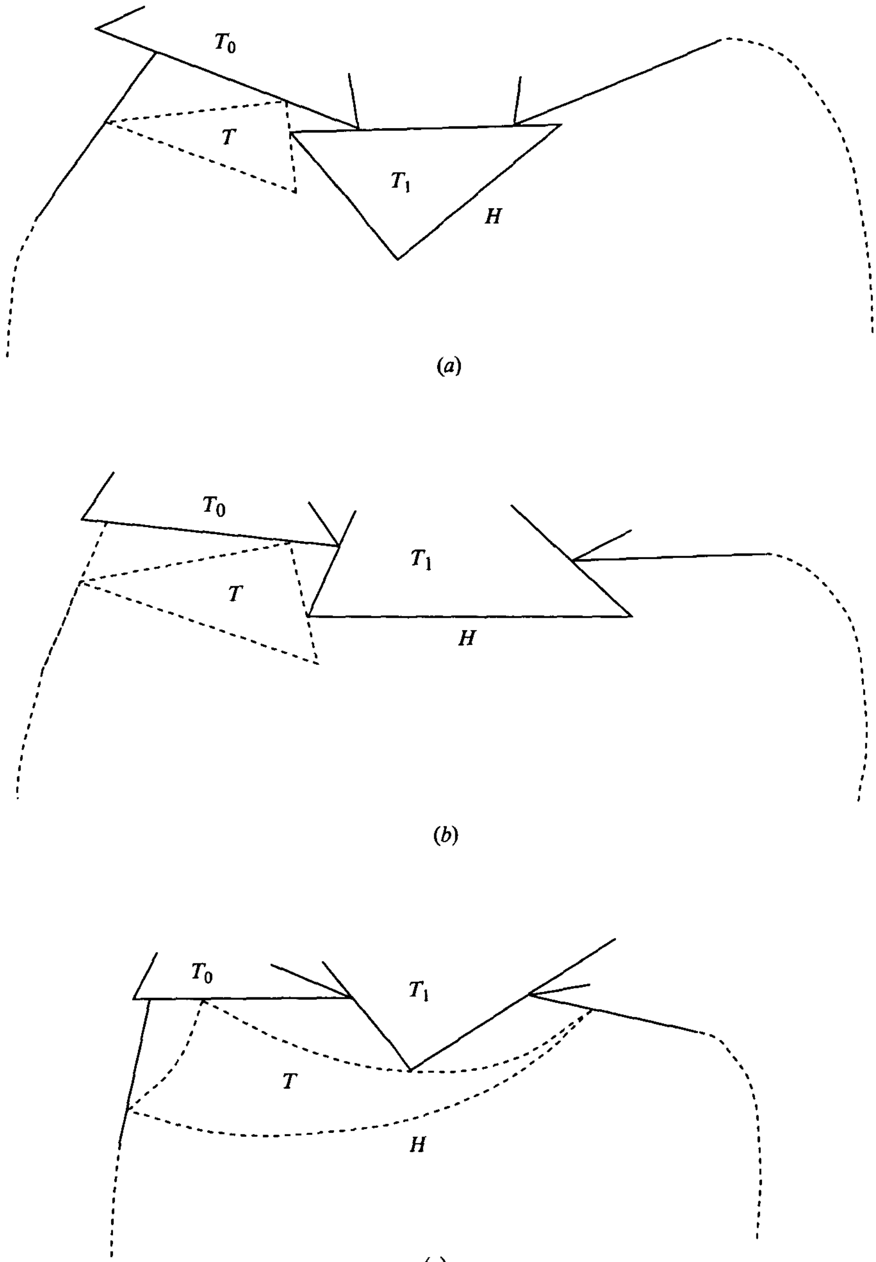


Figure 2 How to fill up the hole H

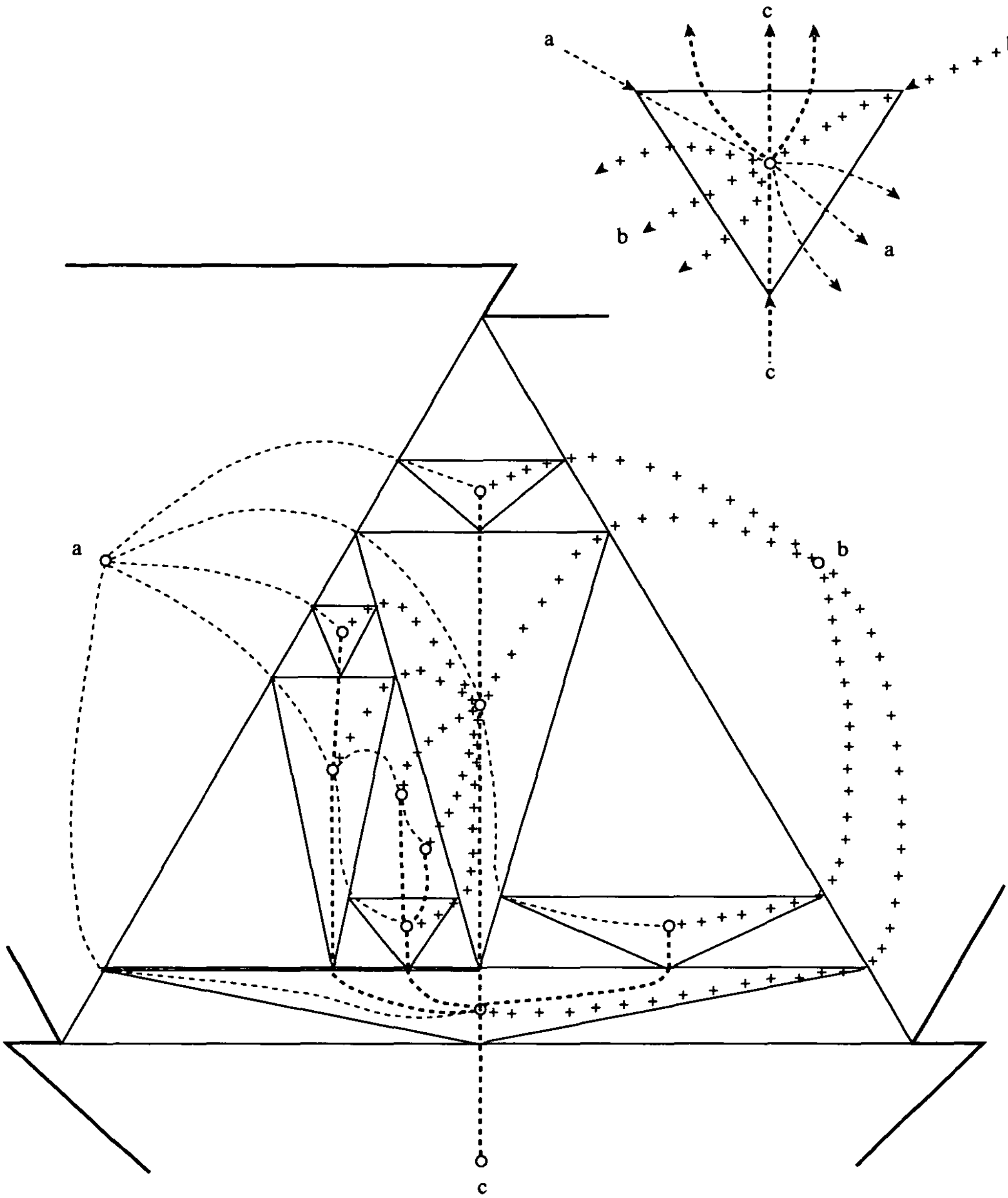


Figure 3 The contact graph edges in three colours

S inside the cycle produces a subsystem of S that is still biconnected. For the hole H so defined, $t_H = k$, and $p_H \geq t_H - 2$, because each triangle except the first and the last have exactly one free node in H. According to Theorem 2.1, the hole H cannot be filled up, and this contradicts the maximality of S. □

Proof of theorem 2.2. Given an internal triangle T, consider two different chains defined by both starting from T. By the argument of the lemma, these two chains cannot end at one and the same triangle. Then the three chains starting at T lead to the three external triangles of S; call them T_r, T_g and T_b, taken in the circular order. The chains ending at a node of T_r (respectively T_g, T_b) constitute an acyclic connected subgraph of $G(S)$, which we call the red (respectively green, blue) tree, and which we orient away from T_r (respectively T_g, T_b). Thus, the edges of $G(S)$ are partitioned into a red, a green and a blue tree, all three oriented away from their respective roots. The nodes belonging to one and the same side all have the same image under f, and hence belong to the same tree. It follows that the sides of each triangle are colored red, green and blue. Now consider three adjacent triangles, and note that the side colors are all in the same circular order. Therefore, all the triangles are colored in the same circular order. The nodes of a triangle represent the three incoming edges; the other contacts, sorted by colors, represent outgoing edges. So, this coloration and this orientation define a Schnyder partition of $G(S)$. □

Remark. Two non-isomorphic triangle contact systems representing the same maximal plane graph G define two distinct Schnyder partitions of G.

Now, given a Schnyder partition of a maximal planar graph, the way to generate a triangle contact system that defines it is the purpose of the next section.

3. Construction of a triangle contact system

In this section we prove the main result of the paper. Let us first recall how a shelling order defines a Schnyder realizer (using the above notation). When at step k the vertex v_k is packed, color red (respectively green) the leftmost (respectively the rightmost) edge incident to v_k and C_{k-1}, and orient it toward v_k; color blue all the other edges incident to v_k and C_{k-1}, and orient them away from v_k. Each vertex of G gets a red and a green father when packed and a blue father corresponding to its last neighbor packed. It is easy to check that this partition into three colored trees is a Schnyder realizer. If we extend the coloration to the outer face in such a way that each edge is assigned to one of the two trees to which it is incident, we get a Schnyder partition.

We shall first prove a representation result for maximal plane graphs. The general case (Theorem 3.2) will follow immediately.

Theorem 3.1. *Any maximal plane graph G has a triangle contact representation.*

Proof. Consider a shelling order and its corresponding Schnyder realizer. In the following,

T_k will denote the triangle representing the vertex v_k and $\phi_r(k)$ (respectively $\phi_g(k), \phi_b(k)$) the shelling label of the red (respectively green and blue) father of v_k in the shelling order.

We start with a maximal triangle contact system formed by the triangles T_1, T_2 and T_n, these triangles having their bases parallel to the x-axis, at ordinates $1, 2$ and n, respectively.

We construct iteratively the representations of the graphs G_k $(2 < k < n)$. Each triangle T_k $(2 < k < n)$ gets its blue base parallel to the x-axis at ordinate k and its opposite node at ordinate $\phi_b(k)$.

At each step, the triangles bounding the representation of G_k will correspond, in the same circular order, to the vertices of C_k, and the intersection of the unbounded hole with the half-plane $y \geq k$ is y-convex (the intersectection with any vertical line is connected).

Assume that the representation of G_{k-1} has been completed according to the previous constraints. By definition of shelling orders, the neighbors $v_{k_1}, \dots, v_{k_p}$ of v_k in G_{k-1} form an interval of C_{k-1}. As v_k is the blue father of the vertices $v_{k_i} (1 < i < p)$, the free nodes of the corresponding triangles have ordinate k. The vertices v_{k_1} and v_{k_p} are, respectively, the red and green fathers of v_k. As noticed above, their blue fathers are packed after v_k, and have a label greater than k. Let x_r (respectively x_g) denote the abscissa of the point of the right (respectively left) side of triangle T_{k_1} (respectively T_{k_p}) at ordinate k. According to the y-convexity of the intersection of the unbounded hole and the half plane $y \geq k - 1$, the region defined by $y > k$ and $x_r \leq x \leq x_g$ is empty. The triangle T_k is placed (crossing free) with coordinates $(x_r, k), (x_b, k), (\alpha_k x_r + (1 - \alpha_k)x_b, \phi_b(k))$ with $\alpha_k \in [0, 1]$. The two conditions on the representation are obviously preserved for G_k.

The representation of G_n is obtained by adding the already defined triangle T_n to the representation of G_{n-1}. □

Remark. We may require the triangles to be isosceles (or right-angle) by a proper choice of T_1, T_2 and T_n and the assignment of the value $\frac{1}{2}$ (respectively 0) to all the α_k coefficients. It is easy to check that there are graphs that cannot be represented by a contact system of equilateral triangles.

Theorem 3.2. *Any plane graph G has a triangle contact representation.*

Proof. The graph G can be augmented into a maximal plane graph G' by adding vertices and edges incident to the added vertices. Then a representation of G follows from a representation of G' by deleting the triangles corresponding to the added vertices. □

Proposition 3.1. *A triangle contact representation of a plane graph G can be computed in $O(n^{2+\epsilon})$ time, with any given $\epsilon > 0$.*

Proof. The graph G can be augmented in linear time into a maximal plane graph G' by adding at most 2 vertices per face. The graph G' and a shelling order of its vertices can be computed in linear time. By choosing a right-angle or isosceles triangle representation, each coordinate needs linear precision. As shown by D. Knuth, each intersection computation may then be achieved in $O(n^{1+\epsilon})$ time. □

In a representation, the ratio between the largest and the smallest triangle may be exponential, and for some graphs this is unavoidable. In the following, we shall describe another type of contact representation on an $n \times n$ grid.

4. Other representations

From the triangle contact representation of a maximal plane graph G, say the isosceles one, we now deduce another representation.

We construct a T-contact system (contact system of T-shaped objects): for each isosceles triangle T draw the perpendicular height corresponding to its horizontal base and, if T is neither T_1, T_2 nor T_n, extend the base of T on both sides, until a contact with the perpendicular height of its red and green fathers is reached (see Figure 4a).

This representation does not require a triangle contact representation to be achieved. Actually, any x-coordinates of the vertical segments compatible with the shelling order and any y-coordinates of the horizontal segments compatible with $O_r \cap \overline{O_g}$ lead to a T-contact representation (see Figure 4b). As a linear extension of a partial order may be computed in linear time, we have the following theorem.

Theorem 4.1. *Any plane graph may be represented by a T-contact system on a $n \times n$ grid in linear time.* □

From an isosceles triangle representation of a maximal plane graph G, one can deduce a representation of G by a contact system of Y-shaped objects (see Figure 5a). Such a representation can be performed directly by using a procedure similar to the one described for triangles, and we can require that the Y-shaped objects be composed of segments belonging to three fixed directions, as shown in Figure 5b.

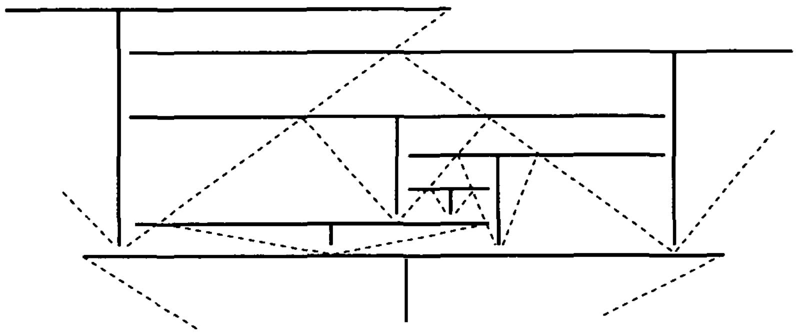
From an isosceles triangle representation of a maximal plane graph, one can also derive a tessellation and a rectilinear representation of G (see Figure 6).

5. Final remarks

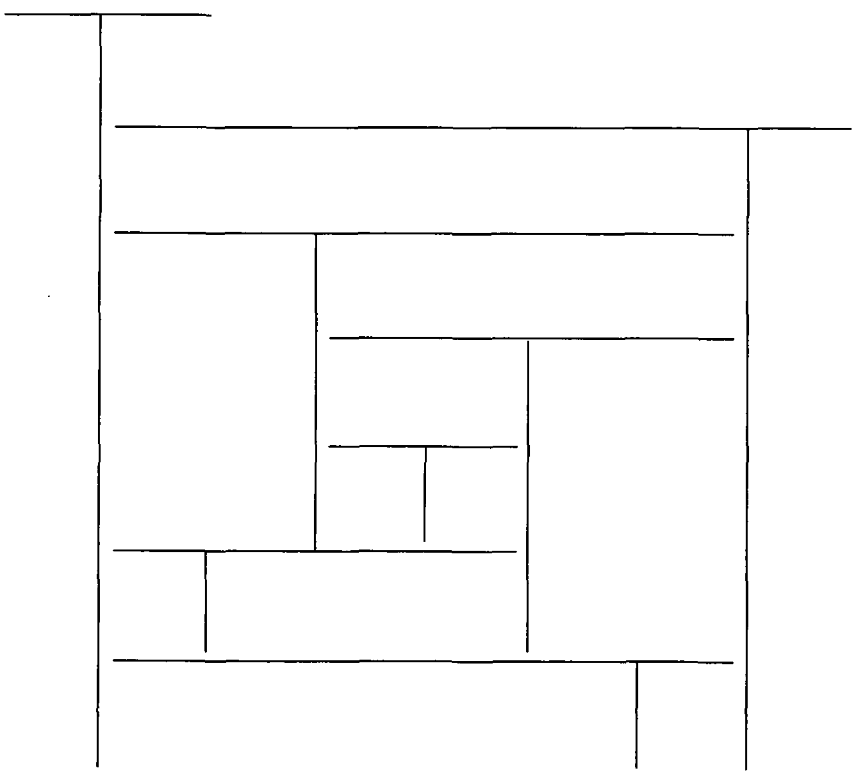
We have shown that two non-isomorphic triangle contact systems representing the same maximal plane graph G define two distinct Schnyder partitions of G. Moreover, we have shown that a Schnyder realizer obtained by a shelling order allows one to construct a maximal triangle contact system with which the realizer is associated. Actually, any Schnyder realizer may be defined by a shelling order. The following theorem is proved in [3].

Theorem 5.1. *Given a Schnyder realizer (Y_r, Y_g, Y_b) of a maximal plane graph G, a total order of $V(G)$ is a shelling order defining (Y_r, Y_g, Y_b) if, and only if, it is a linear extension of $O_r \cap \overline{O_g} \cap O_b$.*

Because of the different possible constructions of the triangles T_1, T_2 and T_n in the previously described algorithm, we have the following theorem.

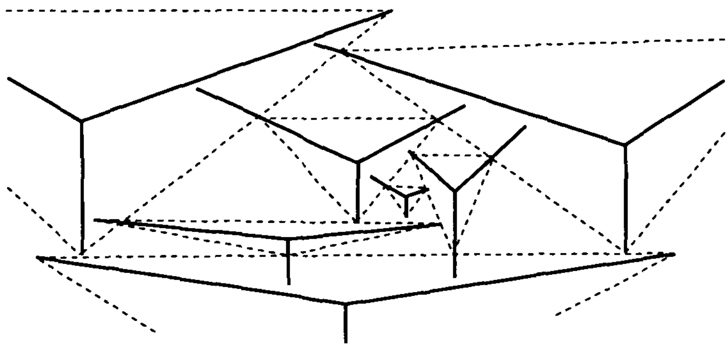


(a)

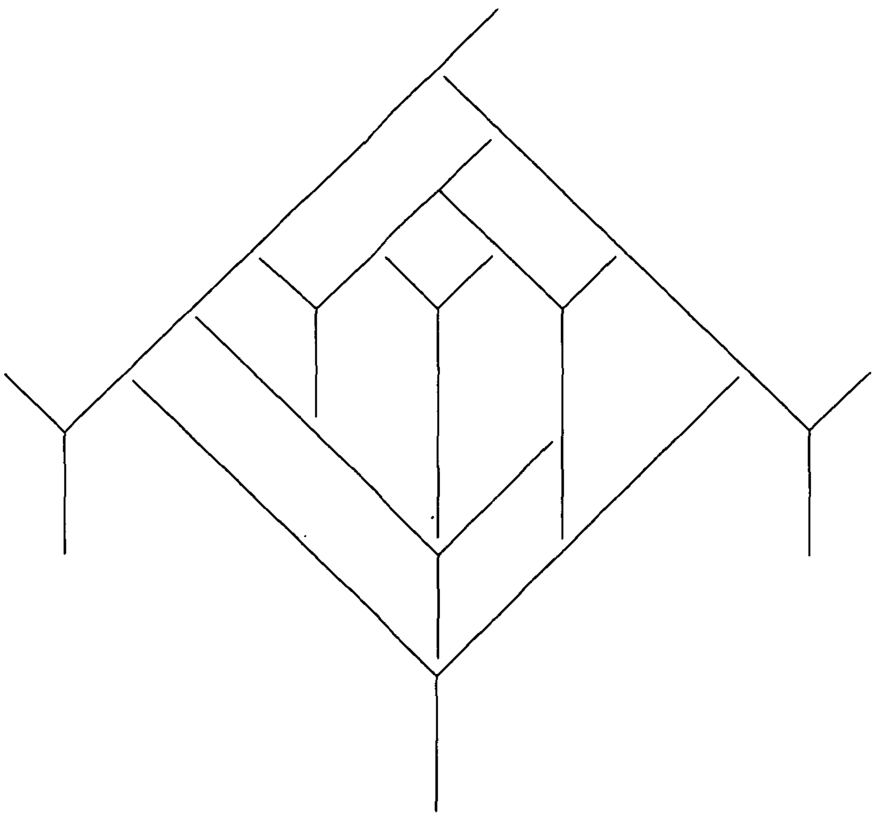


(b)

Figure 4 (a) T-contact (b) T-contact system on the nxn grid



(a)



(b)

Figure 5 (*a*) Y-contact system representation (*b*) Y-contact system with 3 fixed directions on the grid

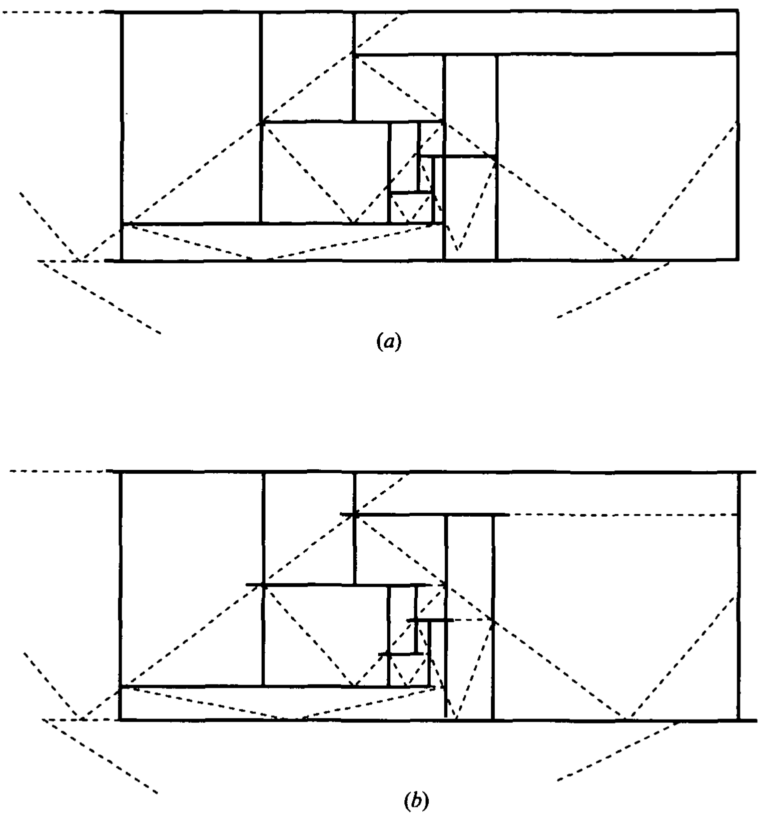


Figure 6 (*a*) Tessellation representation (horizontal segments are vertices, vertical segments are faces and rectangles are edges). (*b*) Rectilinear representation (horizontal segments are vertices, vertical segments are edges)

Theorem 5.2. *The non-isomorphic triangle contact systems representing a maximal plane graph G are in one-to-one correspondence with the Schnyder partitions of G.*

Acknowledgments

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References

[1] Andreev, E. M. (1970) On convex polyhedra in Lobacevskii spaces. *Mat. Sb.* **81** 445–478.

[2] Di Battista, G., Eades, P., Tamassia, R. and Tollis, I. G. (1989) Algorithms for drawing planar graphs: an annotated bibliography. Tech. Rep. No. CS-89-09, Brown University, 1989.

[3] de Fraysseix, H. and de Mendez, P. O. (In preparation) *On tree decompositions and angle marking of planar graphs.*

[4] de Fraysseix, H., de Mendez, P. O. and Pach, J. (submitted) *A streamlined depth-first search algorithm revisited.*

[5] de Fraysseix, H., de Mendez, P. O. and Pach, J. (1993) Representation of planar graphs by segments. *Intuitive Geometry* (to appear).

[6] de Fraysseix, H., Pach, J. and Pollack, R. (1990) Small sets supporting Fary embeddings of planar graphs. *Combinatorica* **10** 41–51.

[7] B. Mohar (To appear) *Circle packings of maps in polynomial time.*

[8] Rosenstiehl, P., and Tarjan, R. E. (1986) Rectilinear planar layout and bipolar orientation of planar graphs. *Discrete and Computational Geometry* **1** 343–353.

[9] W. Schnyder (1990) Embedding planar graphs on the grid. In: *Proc. ACM-SIAM Symp. on Discrete Algorithms* 138–148.

[10] Tamassia, R. and Tollis, I. G. (1989) Tessellation representation of planar graphs. In: *Proc. Twenty-Seventh Annual Allerton Conference on Communication, Control, and Computing* 48–57.