Computing the Equivariant K-theory Class of the Subvariety of Triality-Symmetric Maps

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Notation

- A, B matrices
- V, W are:
  - finite dimensional vector spaces
  - over arbitrary field F
- $v \in V,\ w \in W,\ \alpha \in F$

Linear Transformations

A linear transformation from V into W is a function $T: V \rightarrow W$ which assigns to each vector $v_1, v_2 \in V$ a unique vector $T(v_1), T(v_2) \in W$ such that:

$-T(v_1+v_2)=T(v_1)+T(v_2)$ *additivity*

$-T(\alpha v)=\alpha T(v)$ *homogeneity*

Sets of Linear Transformations

- Let $L(V,W)$ denote the set of all linear transformations of V into W.
  - Let $L(V) = L(V,V)$
- $L(V,W)$ is a vector space over F
- An element of $L(V,W)$ corresponds to some matrix (only true when V,W finite dim)
- $\dim V = n$, $\dim W = m$

  $\Rightarrow \dim L(V,W) = nm$

Dual Space

- $f \in L(V, F)$ is a linear functional
  - Ex: integration of a continuous function,
    evaluation at 0,
    $\text{inner product}: f(\vec{x}) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$
    $$= \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
- **Def:** The vector space of all linear functionals, L(V,F) = V*, is the dual space
- Note: V** = V

Direct Sum: V⊕W

- Let $\{ e_1, \ldots, e_n \}$ be a basis for V
- Let $\{ f_1, \ldots, f_m \}$ be a basis for W
- Then $\{ \tilde{e}_1, \ldots, \tilde{e}_n, \tilde{f}_1, \ldots, \tilde{f}_m \}$ is a basis for $V \oplus W$ where

$$\tilde{e}_i = \begin{pmatrix} e_i \\ 0 \end{pmatrix}$$

$$\tilde{f}_j = \begin{pmatrix} 0 \\ f_j \end{pmatrix}$$

- $v \oplus w = \begin{pmatrix} v \\ w \end{pmatrix}$
- $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$
- $(A \oplus B)(v \oplus w) = \begin{pmatrix} Av \\ Bw \end{pmatrix}$

Tensor Product $V \otimes W$

- Basis for $V \otimes W$ is $\{ e_i \otimes f_j | 1 \leqslant i \leqslant n , 1 \leqslant j \leqslant m \}$
- $v = \sum_{i=1}^{n} \alpha_i e_i$
- $w = \sum_{j=1}^{m} \beta_j f_j$
- $v \otimes w = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j e_i \otimes f_j$

Example

- dim V = 3, dim W = 2
- Basis for $V \otimes W$ is:
  $\{e_1 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_1, e_2 \otimes f_2, e_3 \otimes f_1, e_3 \otimes f_2\}$
- $v = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$
- $w = \beta_1 f_1 + \beta_2 f_2$
- $v \otimes w = \begin{pmatrix} \alpha_1 \beta_1 & \alpha_1 \beta_2 & \alpha_2 \beta_1 & \alpha_2 \beta_2 & \alpha_3 \beta_1 & \alpha_3 \beta_2 \end{pmatrix}^T$

Second Exterior Product

- $\wedge^2 W = \dfrac{W \otimes W}{< w \otimes w | w \in W >}$

- From ex:
  - $\wedge^2 W$ spanned by $\{ f_1 \wedge f_{1,} f_1 \wedge f_{2,} f_2 \wedge f_{1,} f_2 \wedge f_2 \}$
  - $f_1 \wedge f_1 = f_2 \wedge f_2 = 0$
  - $f_2 \wedge f_1 = -f_1 \wedge f_2$

    $\Rightarrow \dim \wedge^2 W = 1$

Consider $Z = L(V, L(V) \oplus V^*)$ with

$\dim V = 2$

==> $\dim Z = 2 \cdot (2 \cdot 2 + 2)$

==> elements of Z are 6x2 matrices

- Let $\phi \in Z = L(V, L(V) \oplus V^*)$
- $\phi : V \rightarrow L(V) \oplus V^*$
- Let $\phi = (\phi_{1,} \phi_2)$ where:
  - $\phi_1 : V \rightarrow L(V)$
  - $\phi_2 : V \rightarrow V^*$

Problem

Find equations for $\Omega \subseteq Z = L(V, L(V) \oplus V^*)$
such that

$\Omega = \{ \phi = (\phi_{1,} \phi_2) \in Z \mid \phi_{1,} \phi_2 \text{ satisfy certain conditions} \}$

Conditions will make ϕ triality symmetric

What do I mean by equations?

- Let $A = (a_{ij})$ be the matrix representation of some ϕ.
- Want to place conditions on the a_{ij}
- $\text{Ex}:$

$$a_{12} = a_{31} + a_{22}$$
$$\Leftrightarrow a_{12} - a_{31} - a_{22} = 0$$

Condition on ϕ_2

- $\phi \in L(V, L(V) \oplus V^*),\ \ \phi : V \to L(V) \oplus V^*$
- $\phi = (\phi_{1,} \phi_2),\ \ \phi_2 : V \to V^*$
- Let $f \in V^*$
- Define $\phi_2^* : V^{**} \to V^*$ such that
  $\phi_2^*(f) = f \circ \phi_2$
  - Note : $\phi_2^* : V \to V^*$
- Condition : $\phi_2 = \phi_2^*$

Condition on ϕ_1

- $\phi_1 : V \to L(V)$
- Consider: $\tilde{\phi}_1 : V \times V \to V$
  - $-V \times V$ the cartesian product
- $\tilde{\phi}_1(v_{1,} v_2) = \phi(v_1)(v_2)$
- Condition: $\tilde{\phi}_1(v_{1,} v_2) \wedge v_3$ invariant under permutations

What Next?

Using equations derived from conditions on ϕ, compute the equivariant K - theory class of the subvariety of maps that are triality symmetric

END