A Cyclic Extension of Heegaard Splittings

Rutgers Math REU Summer 2012
Presentation I

Preliminaries

Definition: A 3-Manifold is a three-dimensional topological space in which each point has a neighborhood that is homeomorphic to an open subset of three-dimensional Euclidean space.

Example: The 3-Sphere, S^3, which is the set of all points in four-dimensional Euclidean space equidistant from a center point.

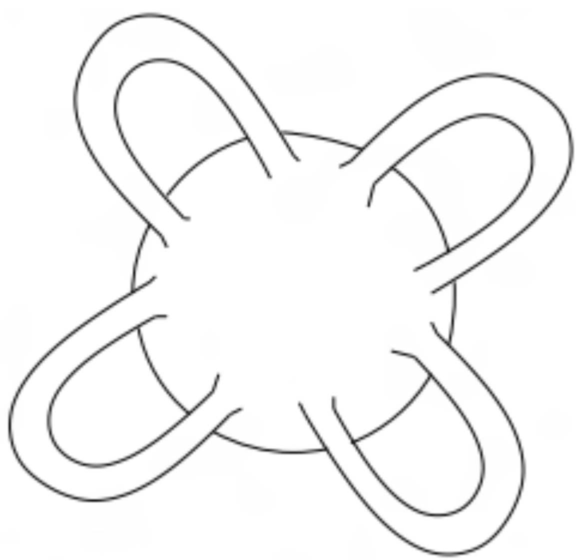
Preliminaries

Definition: A Handlebody is a topological object that is diffeomorphic to a regular neighborhood of a set of circles in $\mathbb{R}^3$ that are connected at a single common point.

If this set contains g circles, we say that the handlebody has genus g.

Preliminaries

Example: A handlebody of genus 4



Preliminaries

Definition: A <u>Heegaard Splitting</u> (or Heegaard Decomposition) of a closed oriented 3-manifold M is a pair of handlebodies H_0 and H_1 such that $M=H_0 \cup H_1$ and $H_0 \cap H_1 = \partial H_0 = \partial H_1$ (here ∂H_i indicates the boundary of H_i).

Alternatively, given two handlebodies H_0 and H_1, we can 'glue' them along their common boundary to get a closed oriented 3-manifold.

Preliminaries

Theorem: Every closed oriented 3-Manifold admits a Heegaard Splitting.

Proof: Let M be such a 3-Manifold and consider a triangulation of M. This induces a three-dimensional graph structure on M. We construct a Heegaard Splitting as follows: Replace each vertex of the triangulation by a ball and each edge by a cylinder. Let H_0 be the union of these objects and let $H_1 = M \setminus H_0$.

Preliminaries

Here we consider smooth functions $f: M^n \to \mathbb{R}$ where M^n is an n-dimensional manifold.

Recall that a point P is a <u>critical point</u> of f if all of the partial derivatives of f vanish at P.

Also recall that the Hessian Matrix of f at P, H (P), is given by the second partial derivatives where $H_{ij}=\partial^2 f/\partial_{xi}\partial_{xj}$.

P is <u>non-degenarate</u> if H(P) is non-singular

Preliminaries

Definition: We say that the function $f: M^n \to \mathbb{R}$ is a Morse Function if all of its critical points are non-degenerate.

Definition: The index of a critical point P is the number of negative eigenvalues in the diagonalization of H(P) with respect to a basis of its eigenvectors.

So a local min has index 0 and a local max has index n

A Morse-Theoretic Approach

Given a closed oriented 3-Manifold M, a morse function f: $M \to \mathbb{R}$ induces a Heegaard Splitting of M. This is because the inverse image of any regular (non-critical) point in the range of f is a two-dimensional surface of genus g and this surface serves as the Heegaard Surface for a Heegaard Splitting.

Moving along the range of f, as we pass through critical points of index 1 or 2, the genus of the inverse image will increase or decrease by 1.

Extension to the Cyclic Case

In this case we map a closed oriented 3-manifold to the Circle S^1.

Such a mapping still induces a splitting, what will be called a Cyclic Heegaard Splitting, but instead of Handlebodies, we will be dealing with so-called 'Compression Bodies'.

Extension to the Cyclic Case

In this Cyclic case, the issue of uniqueness arises:

-Given two Cyclic Heegaard Splittings, in what way(s) are they related?

-Can we get from one to the other with a sequence of well-defined 'moves'?

-For the non-cyclic case it is known that Heegaard Splittings are unique up to <u>Stabilization</u>, or increasing the genus of the Heegaard Surface.   Does there exist an extension of this idea to the Cyclic Case?

References

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[2] **P. Ozsvath, Z. Szabo**, *An introduction to Heegaard Floer homology*, In “Floer homology, gauge theory, and low-dimensional topology” volume 5 of Clay Math. Proc., pages 3–27. Amer. Math. Soc., Providence, RI, 2006.

[3] **N. Savaliev**, *Lectures on the topology of 3-manifolds*. de Gruyter Textbook. Walter de Gruyter & Co., Berlin, 1999.