A Cyclic Extension of Heegaard Splittings

Rutgers Math REU Summer 2012
Presentation II

Introduction

Recall: A Heegaard splitting of a closed oriented 3-manifold M is a pair of handlebodies H_0 and H_1 such that $M=H_0 \cup H_1$ and $H_0 \cap H_1 = \partial H_0 = \partial H_1$

Theorem (Reidemeister-Singer): Let M be a closed orientable 3-Manifold. Then two Heegaard splittings of M become isotopic after suitable stabilizations.

Introduction

We can write this theorem using the language of Morse Theory:

Theorem: Let M be a closed oriented 3-Manifold. Given two ordered Morse functions $f_0, f_1: M \to \mathbb{R}$, there exists a generic path of functions (f_t) from f_0 to f_1 such that for all but finitely many t between 0 and 1 f_t is an ordered Morse Function.

The goal of my research is to establish a similar result for 'circle-valued' Morse functions, i.e. functions $f: M \rightarrow S^1$ which have finitely many critical points, all of which are non-degenerate.

Ideally, one would like a direct analogue of the RS Theorem, simply replacing 'Morse Function' with 'Circle-Valued Morse Function' however it can be seen that this may not be possible. This is because several issues arise when dealing with circle-valued Morse functions that are not present in the usual case.

Some Results

The proof of the usual RS Theorem relies on a result which states that any two Morse Functions (not necessarily ordered) from a 3-Manifold are homotopic via a path of functions, all but finitely many of which are Morse.

In the case of circle-valued Morse Functions, this is not the case because we may need to deal with k-to-1 maps to S^1.

<u>Theorem</u>: Let M be a closed oriented 3-Manifold and let f: $M \to S^1$ be an indefinite circle-valued Morse Function. If the inverse image under f of a point in S^1 is nonempty and has k connected components, then there exists a function g: $M \to S^1$ such that $f = g^k$.

<u>Idea of Proof</u>: We fix a regular point r introduce a Riemannian metric and unit speed gradient on M in order to construct a path from path from r that is closed and connected except for possibly at finitely many points.

This gradient induces a flow function on the path that twists around the manifold k times at unit speed such that at time 1/k we are one fiber 'above' our starting point. So a k-times composition of this particular flow will give us our initial f and thus can serve as g.

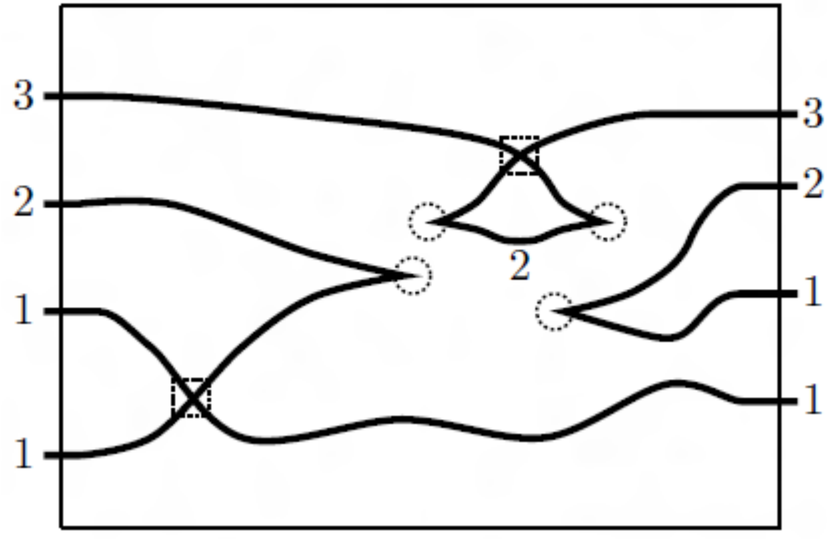
The good thing about this theorem is that it allows us to construct from a function with disconnected fibers, one with connected fibers. Hence, we need only consider such Heegaard splittings with connected fibers.

Another issue that arises in generalizing the RS Theorem is that the standard version of the theorem deals only with stabilization, however with Circle-Valued Morse Functions we need to introduce a new type of move called 'winding'.

The issue of winding arises when considering a homotopy of one circle-valued Morse Function to another. This is because the critical points of circle-valued Morse functions can behave differently.

Cerf Theory

'Cerf Theory' tells us about how critical points of Morse functions behave under a homotopy. This behavior can be visualized with a Cerf Graphic:



In the linear case, critical points can be mapped straight 'across' to other critical points, however in the cyclic case it may be that a critical point winds around this circle under a homotopy

We can easily modify a Cerf graphic to accommodate for this new type of behavior, but what's more important is that winding makes it a bit more difficult to establish a homotopy between two Morse functions or Heegaard splittings.

Critical point winding is significant because it changes the genus of the fibers of the manifold, requiring us to take more care when finding a homotopy between two circle-valued Morse Functions.

Now instead of just using stabilization, we need to be able to correct for genus changes brought about by winding.

<u>Conjecture</u>: The total number of winds of index 1 critical points is equal to the total number for index 2 critical points.

Things for Future Consideration

Given two ordered indefinite circle-valued Morse Functions on the same 3-Manifold, under what conditions does there exist a homotopy between them which involves no critical point winding?

Are we able to assign a 'winding number' to each critical point?

References

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