Research Question Presentation on the Edge Clique Covers of a Complete Multipartite Graph

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Vertex Clique Covers and Edge Clique Covers:

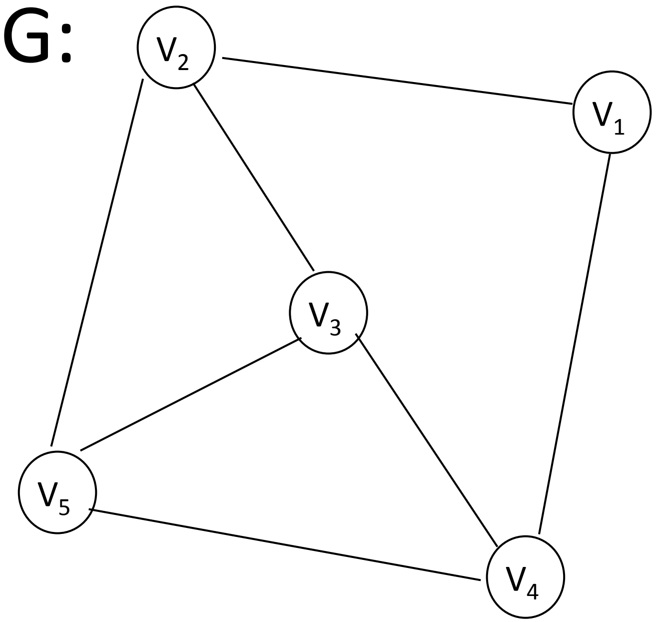
Suppose we have a graph G.

A *clique* S of G is a subset of V(G) such that any two vertices of S are adjacent in G.

A *vertex clique cover* of G is a family of cliques such that any arbitrary vertex of G is contained in some clique in the family.

Example:

G:

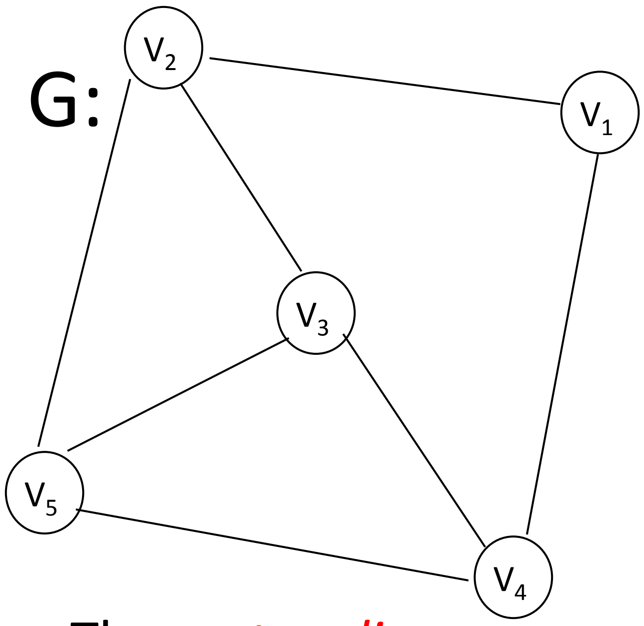


Some vertex clique covers of G:

1) $\{ (V_1, V_4), (V_2, V_3, V_5)\}$
2) $\{ (V_1, V_2), (V_3, V_4, V_5)\}$
3) $\{ (V_1, V_2), (V_4, V_5), (V_3)\}$
4) $\{(V_2, V_3, V_5), (V_3, V_4, V_5), (V_1)\}$

An *edge clique cover* of G is a family of cliques such that any arbitrary edge of G is covered by some clique in the family.
(if an edge is covered by a clique, both of its incident vertices are present in that clique, as cliques only contain vertices and not edges)
Example:

G:



Some edge clique covers of G:

1. $\{ (V_2, V_3, V_5), (V_3, V_4, V_5), (V_1\ V_2), (V_1, V_4)\}$
2. $\{ (V_2, V_3, V_5), (V_3, V_4), (V_4, V_5), (V_1\ V_2), (V_1\ V_4)\}$
3. $\{(V_3, V_4, V_5), (V_2, V_3), (V_2, V_5), (V_1, V_2), (V_1, V_4),$

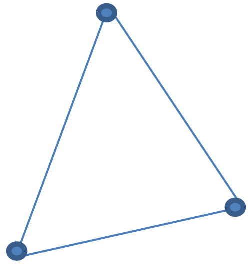
The *vertex clique cover number* of a graph G, denoted θ_v(G), is the minimum size of vertex clique covers of G.
The *edge clique cover number* of a graph G, denoted θ_e(G), is the minimum size of edge clique covers of G.

Complete Multipartite Graphs:

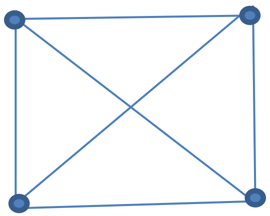
A *complete graph,* denoted K_n, is a graph G with n vertices where every vertex of G is connected to every other vertex.

Examples:

K_3:

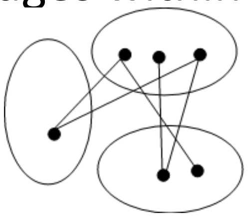


K_4:



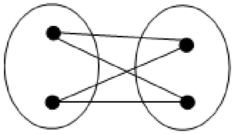
A *multipartite graph* is a graph G that is composed of partitions of V(G) where there are no edges within each partition.

Example:



A *complete multipartite graph* is a multipartite graph where every vertex in each partite site is connected to every other vertex in all the other partite sets.

Example:



If the amount of vertices in each partition differ, then we denote the complete multipartite graph on r partite sets as $K_{n_1,n_2,\ldots\ldots n_r}$, where $n_1 \geq n_2 \geq \ldots\ldots \geq n_r$.

If $n_1 = n_2 = \ldots\ldots = n_r$, then we denote the complete multipartite graph on r partite sets with n vertices in each set as K_n^r. A vertex in K_n^r is denoted as v_j^i, where $1 \leq i \leq r$ and $1 \leq j \leq n$.

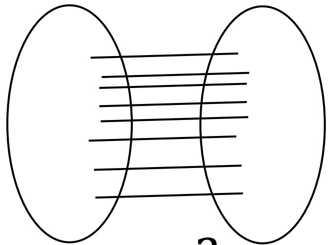
Note that $\theta_v(K_{n_1,n_2,.....n_r})$=$n_1$is a known result, where $n_1 \geq n_2 \geq ...... \geq n_r$.

Proof: Each vertex in the largest set, n_1, cannot be connected with another vertex in n_1 by definition, so they must all be contained in distinct cliques.
Thus, $\theta_v(K_{n_1,n_2,.....n_r}) \geq n_1$.
Compose cliques as such:
for each $1 \leq i \leq n_1$, let $S_i = \{v_i^j \mid 1 \leq j \leq r\}$.
Then $\{S_1, S_2, \dots . S_n\}$ is an edge clique cover of $K_{n_1,n_2,.....n_r}$.
(for example, S_1=$\{v_1^1$, v_1^2,..., $v_1^r\}$, S_2=$\{v_2^1$, v_2^2,..., $v_2^r\}$,......, S_{n_1}=$\{v_{n_1}^1$, $v_{n_1}^2$,..., $v_{n_1}^r\}$)
This process illustrates that $\theta_v(K_{n_1,n_2,.....n_r}) \leq n_1$.
$\therefore \theta_v(K_{n_1,n_2,.....n_r}) = n_1$ ■

For this project, we would like to focus on $\theta_e(K_{n_1,n_2,.....n_r})$.
Determining the edge clique cover number of a complete multipartite graph is a much more difficult task.
It is well known that $\theta_e(k_{n_1,n_2,.....n_r}) \geq n_1 n_2$. However, we wish to know a more precise value of $\theta_e(k_{n_1,n_2,.....n_r})$. We look at several cases for the value of r, where $n_1 \geq n_2 \geq ...... \geq n_r$.

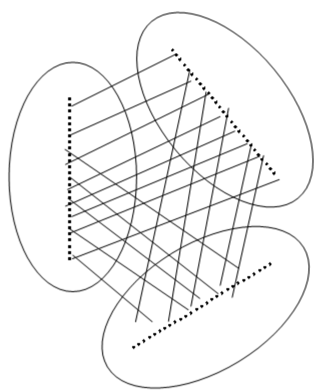
Several known results:

1) $\theta_e(k_n^2) = n^2$



there are a total of n^2 edges in this graph and each edge belongs to a unique clique since there are no other connections other than the edges from one partite set to the other.

2) $\theta_e(k_n^3) = n^2$



3) When n is a power of a prime, $\theta_e(k_n^4) = n^2$

Example:

Suppose $n = 4$ and $r = 4$. We choose our cliques as such:

$\{v_1^1, v_1^2, v_1^3, v_1^4\}$ $\{v_2^1, v_1^2, v_3^3, v_2^4\}$ $\{v_3^1, v_1^2, v_4^3, v_3^4\}$ $\{v_4^1, v_1^2, v_2^3, v_4^4\}$
$\{v_1^1, v_2^2, v_2^3, v_2^4\}$ $\{v_2^1, v_2^2, v_4^3, v_1^4\}$ $\{v_3^1, v_2^2, v_3^3, v_4^4\}$ $\{v_4^1, v_2^2, v_1^3, v_3^4\}$
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$\{v_1^1, v_4^2, v_4^3, v_4^4\}$ $\{v_2^1, v_4^2, v_2^3, v_3^4\}$ $\{v_3^1, v_4^2, v_1^3, v_2^4\}$ $\{v_4^1, v_4^2, v_3^3, v_1^4\}$

Any edge of k_4^4 is covered by one of the above 16 cliques. Thus, it is a minimum edge clique cover of K_4^4.

A *Latin Square* of order n is an $n \times n$ rectangular array where each row and column includes all the integers from 1 to n. We denote the entry in the i^{th} row and j^{th} column, where $i, j \in \{1,2, \dots, n\}$, as $L(i, j)$.

Two Latin Squares L_1 and L_2 are said to be *orthogonal* if for any $i, j \in \{1,2, \dots, n\}$, $\exists\, k, l \in \{1,2, \dots, n\}$ such that $L_1(k, l) = i$ and $L_2(k, l) = j$.

Example: Suppose $n = 5$. Then the following two Latin squares are orthogonal.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

$L(n)$ is defined to be the maximum size of mutually orthogonal Latin Squares.

It is known that for any integer n, $L(n) \leq n-1$ and if n is a prime power, then $L(n) = n-1$.

If $m \leq L(n)+2$, where m is the amount of sets of a complete multipartite graph, then we can construct an edge clique cover for k_n^m by looking at our set of mutually orthogonal Latin Squares. Each clique has m entries where $i, j \in \{1,2, \ldots.., n\}$ are the first and second entries in the clique, $L_1(i,j)$ is the third,……, and $L_{n-1}(i,j)$ is the m^{th} entry. We will end up with a total of n^2 such cliques since there are a total of n^2 ordered pairs (i,j).

For example, we can come up with $\theta_e(k_n^3)$ by examining a Latin Square of order n.

1	2	3	……	n
2	3	……	n	1
3	……	n	1	2
……	n	1	2	3
n	1	2	3	……

For the first n cliques, we observe all $1,j$ values in the first row.
We find cliques similarly for values in all n rows.

{1,1,1}	{2,1,2}	{3,1,3	……….	{n,1,n}
{1,2,2}	{2,2,3}	:	……….	{n,2,1}
{1,3,3}	:	{3,n-2,n}	……….	{n,3,2}
:	{2,n-1,n}	{3,n-1,1}	……….	{n,4,3}
{1,n,n}	{2,n,1}	{3,n,2}	……….	:

Now suppose we have two orthogonal Latin Squares, L_1 and L_2, of order 5.

L_1

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

L_2

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

Since we have two orthogonal Latin Squares, $m \leq 2 + 2$=4.

So we can find $\theta_e(k_n^4)$ by forming the following cliques:

Row i Column j $L_1(i,j)$ $L_2(i,j)$

{1,1,1,1}, {1,2,2,2}, {1,3,3,3}, {1,4,4,4}, {1,5,5,5}, {2,1,2,3}, {2,2,3,4}, {2,3,4,5}, {2,4,5,1}, {2,5,1,2}, etc.

Since there are n^2 entries in the Latin Squares, and we form another clique for each entry, we have $\theta_e(k_n^4)$ = n^2

In this project, we will focus on $\theta_e(K_{n_1,n_2,....,n_r})$.

Research questions on $\theta_e(K_{n_1,n_2,....,n_r})$:

1) Suppose $n_1 = n_2 =.....= n_r$. Can we find better bounds for the edge clique cover number of any complete multi-partite graph K_n^m by looking at sets of mutually orthogonal Latin Squares?
2) More specifically, what are the values of $\theta_e(k_n^3)$or $\theta_e(k_n^4)$?
3) What conditions can we put on $n_1, n_{2,}......, n_r$ so that $\theta_e(K_{n_1,n_2,....,n_r})$=$n_1 n_2$?

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