A Trip to Prague: What I Have Learned

Marissa Loving

August 7, 2012

These past days in the Czech Republic have been some of the busiest and most exciting in my life. From morning seminars to afternoon romps through castles and gardens to evenings spent absorbing the local food and culture, not a minute has gone to waste. As an aspiring mathematician it has been a great honor to meet and speak to mathematicians of such excellence and importance as Jaroslav Nešetřil, Pavel Valtr, Jiří Fiala, and many others.

I really enjoyed our first week of lectures and problem solving. Although I found each of the lectures stimulating, I was particularly excited by the lecture given by Yared Nigussie on W.Q.O. (Well-Quasi Ordering) Theory. I had never found myself intrigued by the topics I had encountered in order theory, but this talk gave me a new perspective.

Yared began by introducing us to the concept of a quasi-order. The idea of a quasi-order is similar to that of a partial order, which I was familiar with from some of my computer science classes. Much like a partial order can be thought of as a more relaxed form of a total order, that is, it allows us to have some notion of *greater than* or *less than* in a set where not necessarily all elements are related to each other, a quasi-order can be thought of, intuitively, as some more relaxed or general form of a partial order.

Definition: A <u>quasi order</u> $(Q, \leq)$ consists of a set Q and a relation $\leq$ such that $\leq$ is both reflexive and transitive.

Thus, even the definition is very similar to that of a partial order except it omits the need for $\leq$ to be antisymmetric. Note that, given this definition. It is possible for two distinct elements, $q, q' \in Q$ to have the property that $q \leq q'$ and $q' \leq q$.

With this mathematical machinery in place, Yared returned to the familiar concept of the Well-Ordering Principle with this question in mind, "Does a similar concept hold for quasi-ordered sets?" It turns out that there are some pretty neat results proved about Well-Quasi Ordered sets. However, before we were able to discuss some of these results we needed to develop a notion of what it means to be not just quasi-ordered, but *well* quasi-ordered.

Definition: A quasi-order $(Q, \leq)$ is said to be <u>well quasi-ordered</u> if for every infinite sequence $q_1, q_2, q_3, \ldots$ you will always find $i < j$ such that $q_i \leq q_j$.

So what is this *similar concept* that was mentioned? We consider Higman's Lemma as well as a theorem by Kruskal. The versions introduced to us by Yared are given below.

<u>Higman:</u> If Q is well quasi-ordered, then $Q^{<\omega}$ is also well quasi-ordered under "embedding".

<u>Kruskal:</u> Trees under embedding are well quasi-ordered.

Although the original proof of Higman's Lemma was very long and complicated, Nash Williams gave a very simple and elegant proof of this using an argument known as the minimal-bad-sequence argument. The proof uses three infinite "tapes", a constructive approach, and a series of contradictions to arrive at the desired conclusion. Yared went over Nash Williams' proof with us and I was excited to be able to follow along quite well.

This was not always the case during the Midsummer Combinatorics Workshop the following week because of the select audience the talks were often given at such a high level that I found myself quite lost after the first few definitions. Nonetheless, it was wonderful to be immersed in such an intense mathematical environment and I found myself eagerly listening to talks even when I had very little of the mathematical mechanics to understand the more intricate details of the concepts and proofs being presented.

Several of the talks which stood out to me were Dhruv Mubayi's talk, *Quasi-random Hypergraphs*, Tomas Gavenčiak's presentation, *Hyper-tree-depth and Hypergraph Pairs*, and Kolja Knauer's discussion, *Simple Treewidth.* However, I think I most enjoyed Jan Hubicka's talk entitled *Order of Locally Constrained Homomorphisms.* This may be attributable, in part, to the fact that I actually knew much of the vocabulary that was being employed. I also thought it was really neat to see the work of Nešetřil being cited in his own workshop.

Jan's presentation began with an introduction to the set, G, of graph homomorphisms on a graph with the identity homomorphism as a relation $\leq$ form a quasi-order $(G, \leq)$. If we consider to such sets G and H we can say that G and H are hom-equivalent if and only if $G \leq H \leq G$. Now, what if we consider the set of functions that are only locally surjective, injective, or bijective, respectively? It turns out that these locally injective, surjective, and bijective homomorphism induce a partial ordering and we can, in fact, prove the following lemma.

<u>Lemma:</u> Every locally surjective or bijective homomorphism $F : G \to H$ is surjective when H is connected.

We can use the following result by Nešetřil

<u>Theorem:</u> Every locally injective homomorphism $F : G \to G$ is an automorphism of G for connected graphs G.

to prove that if $drm(G) = drm(H)$, then every locally injective or surjective homomorphism is locally bijective for connected graphs G where drm is the degree refinement matrix.

So, in some sense, it seems that the degree refinement matrix describes or at least distinguishes the equivalence classes of the equi-parititions of our graph.

Jan finished by noting that the homomorphism order is universal on partial orders and lattices and asking the audience to consider the case of duality pairs. This was how many of the speakers ended their talks, not with this specific question, but simply with a series of questions, conjectures, or open problems, which lent itself to the overall flexible, hands-on feeling of the workshop. An environment that I found very exciting because it helped me envision myself in a similar setting as an actual mathematician and researcher.

Although I was not able to work on any of the questions that were posed during the Midsummer Combinatorics Workshop, the speakers during the first week of our stay in Prague did provide all of us students with some problems to tackle, which were quite interesting as they involved topics, such as graph theory, that I had never worked out proofs for.

Problem Set

1. For every natural number $n \geq 2$ there is a cograph of order n that is not an equivalence graph.

> *Proof:* Consider the complete graph of order n for any $n \in \mathbb{N}$, that is K_n. First note that K_n is a cograph for every n. Now, suppose we delete a single edge of K_n, to form a graph we will call K_n'. K_n' is still a cograph. However, K_n' is no longer an equivalence graph becasuse their exist two vertices of K_n' that are connected by a path, yet no edge exists between them. Hence, for every $n \in \mathbb{N}$ there exists a cograph of order n that is not an equivalence graph.

2. A graph is perfect if the chromatic number of each induced subgraph equals the clique number of the induced subgraph.

(a) Give an example of a graph that is not perfect.

Any odd numbered cycle.

(b) Show that all equivalence graphs are perfect.

Proof: Let G be an equivalence graph of n vertices. Note that every induced subgraph of an equivalence graph is also an equivalence graph. Thus, the chromatic number of every induced subgraph will be equal to the clique number of that subgraph. Namely, a subgraph of order m will also have chromatic and clique number m since it is an equivalence graph. Hence, G is perfect and we can conclude that every equivalence graph is perfect.

(c) Show that the union of two vertex-disjoint perfect graphs is perfect.

Proof: Let G and H be two vertex-disjoint perfect graphs. Consider an induced subgraph I of $G \cup H$. We have three cases: all the vertices of I are contained in G, all the vertices of I are contained in H, or the vertices of I are contained in both G and H.

If all the vertices of I are contained in G, then I is an induced subgraph of the perfect graph G and so its clique number is equal to its chromatic number, trivially. Without loss of generality, this argument can be applied to I when all of its vertices are contained in H.

If the vertices of I are contained in both G and H and these are vertex-disjoint graphs then the vertices of I contained in G and those contained in H are completely disjoint sets. Thus, I is the vertext-disjoint union of two induced subgraphs, one of G and one of H. Becuse G and H are vertex disjoint it is clear that the clique number of I will be equal to the maximum of the clique numbers of the two subgraphs induced in G and H, respectively, by I. Similarly, the chromatic number of I will be equal to the maximum of the chromatic numbers of the two subgraphs induced in G and H, respectively, by I. Since both G and H are perfect graphs, their induced subgraphs will each have clique number equal to chromatic number and we can conclude that the clique number of I

will be equal to the chromatic number of I. Hence, the vertex-disjoint union of two equivalence graphs and thus, an equivalence graph.

Hence, in any case, every subgraph I of $G \cup H$ is an equivalence graph and we can conclude that the vertex-disjoint union of perfect graphs is perfect.

3. Determine the tree width of all outer planar graphs.

Proposition: The tree width of any outer planar graph is at most 2.

Proof: Let G be an outer planar graph. Consider the graph G' created by adding edges to G until we have attained a maximal outer planar graph. Consider the reduced dual, T, of G', that is the dual of G' minus the vertice for the outer plane. We claim that T is a tree decomposition of G'.

First, we will show that T is a tree by showing that it does not contain a cycle. Suppose to the contrary that T contains a cycle. If T contains a cycle of length n then their exists at least one vertex in G' that is completely bounded by faces of G'. However, G' is an outer planar graph and therefore cannot contain an internal vertex, a contradiction. Thus, we conclude that T is indeed a tree.

Second, we will show that T is indeed a tree decomposition of G' because:

- every node of T is a subset of V'_G, since every node of T consists of the edges and vertices bounding a plane region of G';
- by definition every edge of G' is contained in a node of T;
- and we know that every $v \in V_G$ is contained in at least a single node of T and, additionally, that if v appears in mutliple nodes of T these nodes will be adjacent, by construction, and we can conclude for every $v \in V'_G$ the nodes containing v will induce nonempty and connected subtrees in T.

So T has width 2 because $f(t) = 3$ for ever $t \in V_T$. Thus, we have found a tree decomposition of G' with width 2 and can conclude that the tree width of G' is at most 2. Becaue we can obtain our graph G by simply deleting edges from G' we know that G is a minor of G'. Thus, the tree width of G' forms an upper bound on the tree width of G. Hence, the tree width of any outer planar graph is at most 2.

4. Given graph G, the girth of G, denoted $g(G)$, is the number of edges in the smallest induced cycle in G. If G is a cograph, what is the largest possible value of $g(G)$? What is the smallest?

Proof: Consider a cograph G. Note that, for every n P_n is a subgraph of C_n. In fact, it is clear that for every n $P_1, P_2, \ldots, P_{n-1}$ are induced subgraphs of C_n. Thus, since G is a cograph and P_4 free we can conclude that it does not contain an induced subgraph of C_n for $n \geq 5$. Hence, $g(G)$ is at most 4. Trivially, consider the smallest cycle, C_3. Note that C_3 is indeed a cograph with girth 3. Hence, it is clear that $g(G)$ is no less than 3 for any cograph G.

5. Consider the set of bipartite graphs. Is it true that every bipartite graph is a cograph? What if we restrict ousrelves to the set of complete bipartite graphs?

Proof: Note that every every cycle of even length is a bipartite graph. However, C_4 contains

Suppose we look instead at only complete bipartite graphs. Consider a complete bipartite graph G that is formed by joining the vertex disjoint, empty graphs U and W and a collection ω of 4 vertices in V_G. We have three cases, either $\omega \subset V_U$, $\omega \subset V_W$, or $\omega \subset V_U \cup V_W$.

If $\omega \subset V_U$, then the subgraph induced by ω is an empty graph of 4 vertices. Without loss of generality the same holds for $\omega \subset V_W$.

If $\omega \subset V_U \cup V_W$, then we have four cases, but these can be reduced to two without loss of generality. Note that either one vertext in ω lie in U and three in V or two vertices lie in both U and V.

In the first case, since G is a complete bipartite graph, then we have the induced subgraph $K_{1,3}$ which does not contain P_4 as an induced subgraph.

Similarly, in the second case, since G is a complete bipartite graph, then we have the induced subgraph $K_{2,2}$, which once again does not contain P_4 as an induced subgraph.

Therefore, there is no collection of 4 vertices that induce P_4 and we conclude that every complete bipartite graph is indeed a cograph.