# Random Geometric Graphs: Lecture Notes II

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## 1 Existence of a giant component

In this lecture we work in *supercritical regime*, i.e. the density $\lambda$ is always greater than $\lambda_c$. The Poisson Point Process (P.P.P.) is done on a box $B(s) = \left[-\frac{s}{2}, \frac{s}{2}\right]^d \subseteq \mathbb{R}^d$.

Let $B = \prod_{k=1}^{d} [a_k, b_k]$ and let $\pi_k : \mathbb{R}^d \to \mathbb{R}$ be a projection onto the $k$-th coordinate. The box $B$ is *$k$-crossing* for random geometric graph $G(X, r)$ if there are vertices $x'$ and $x''$ such that $|\pi_k(x') - a_k| < \frac{r}{2}$, $|\pi_k(x'') - b_k| < \frac{r}{2}$ and both vertices lie in the same component of $G$.

If for every $k$ the box is $k$-crossing we say that the box is *crossing*.

We restrict ourselves only to the case $d = 2$. We define a rectangular box for a given height $a$: $B(a, j) = [0, ja] \times [0, a]$ (for $j = 1, 2, 4$). Let $\mathrm{LR}_a$ denote the event that there is a 1-crossing component for $B(a, 2)$ on $G(H_{\lambda,s} \cap B(a, 2), 1)$, i.e. the graph is crossing the rectangle from left to right along the longer side. Similarly let $\mathrm{LLR}_a$ be the event that there is a 1-crossing component for $B(a, 4)$ on $G(H_{\lambda,s} \cap B(a, 4), 1)$ and let $\mathrm{SLR}_a$ be the event that there is a 1-crossing component for $B(a, 1)$ on $G(H_{\lambda,s} \cap [0, a] \times B(a, 1), 1)$.

**Lemma. 1** (10.4)**.** *For $d = 2$ $\Pr(\mathrm{SLR}_a) \uparrow 1$ and $\Pr(\mathrm{LR}_a) \uparrow 1$ as $a \uparrow \infty$.*

**Lemma. 2.** *Let $d = 2$ and $\lambda > \lambda_c$, then there exist constants $a_1$ and $c$ such that $Pr\left[LR_a\right] \geq 1 - e^{-ca}$ for all $a \geq a_1$.*

*Proof.* Let $\delta < 1$ be some real number. Consider the box $H_i = [ib, (i + 2)\, b] \times [0, b]$ for $i = 0, 1, 2$. We choose $b > 0$ such that $Pr\left[LR_b\right] \geq 1 - \frac{\delta}{25}$ and $Pr\left[SLR_b\right] \geq 1 - \frac{\delta}{25}$ for $\delta$ (we know that such $b$ exists from the previous lemma). Let $V_1 = H_0 \cap H_1$ and $V_2 = H_1 \cap H_2$.

The occurrence of a vertical crossing in $V_1$ and $V_2$ together with horizontal crossings in $H_0$, $H_1$ and $H_2$ imply the event $LLR_b$. See Figure 1. Thus the probability that $LLR_b$ does not happen is, by Boole‘s inequality,

$$1 - Pr\left[LLR_b\right] \leq \frac{5\delta}{25} = \frac{\delta}{5}.$$

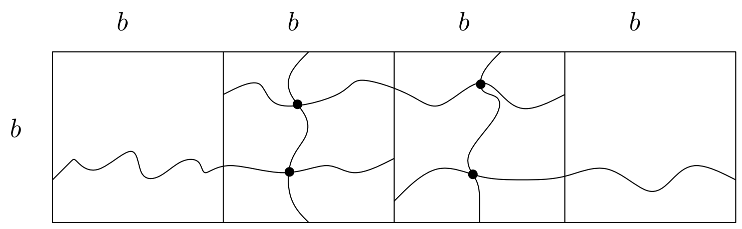


Figure 1: Occurence of $LLR_b$

Since $B(2b,2)$ is a disjoint union of two copies of $B(b,4)$, by independence we get

$$1 - Pr\left[LR_{2b}\right] \leq \left(1 - Pr\left[LLR_b\right]\right)^2 \leq \frac{\delta^2}{25}$$

and similarly for $B(2b,1)$ and $B(b,2)$ we have

$$1 - Pr\left[SLR_{2b}\right] \leq \left(\frac{\delta}{25}\right)^2 \leq \frac{\delta^2}{25}.$$

Using induction we obtain

$$1 - Pr\left[LR_{2^k b}\right] \leq \frac{\delta^{2^k}}{25}$$

and

$$1 - Pr\left[SLR_{2^k b}\right] \leq \frac{\delta^{2^k}}{25}$$

for every positive integer $k$. By the previous lemma there exists $a_0$ such that $Pr\left[LR_a\right] \geq \frac{49}{50}$ and $Pr\left[SLR_a\right] \geq \frac{49}{50}$ for every $a \geq a_0$. For $a > a_0$ we choose $\delta = \frac{1}{2}$ and an integer $m$ such that $2^m a_0 \leq a < 2^{m+1} a_0$. Afterwards we have

$$1 - Pr\left[LR_{2^m b}\right] \leq \frac{2^{-2^m}}{25} < \frac{2^{\frac{-a}{2a_0}}}{25} = e^{-ca}$$

and

$$1 - Pr\left[SLR_{2^m b}\right] \leq \frac{2^{-2^m}}{25} < \frac{2^{\frac{-a}{2a_0}}}{25} = e^{-ca}.$$

$\square$

We now generalize to the space $\mathbb{R}^d$ for $d \geq 2$. Let us denote $c_\infty$ the only infinite component of a graph (we are still in supercritical regime).

For nonnegative $s$ let $c(B(s))$ be $c_\infty \cap B(s)$. We set $\Phi_s = \log^2 s$. Following holds by an ergodic theorem (10.34):

$$\frac{1}{s^d}c(B(s)) \to \lambda p_\infty(\lambda) \mathrm{as} s \to \infty$$

Component $c_\infty$ can be divided into more than part in the box $B(s)$. Let us denote the parts $c_1, c_2, \dots c_n$ (in decreasing order). We can see that on the margin of the box $B(s)$, precisely inside $B(s) \setminus B(s-2)$, there is at least one point $x_i$ belonging to $c_i$ for every $i$ from 1 to $n$ (see Figure TODO). The balls $Ball(x_i, 0.5)$ are disjoint. It holds

$$n \cdot \mathrm{Vol}(Ball) \leq \mathrm{Vol}(B(s) \setminus B(s-2)) = k_1 \cdot s^{d-1}$$

for a constant $k_1$. Thus the number of components $n \leq k_2 \cdot s^{d-1}$ for a constant $k_2$.

We want to prove

$$\frac{1}{s^d}\sum_{i=1}^{n} |c_i| \to \lambda p_\infty(\lambda)$$

We have

$$\lim_{s \uparrow \infty} \Pr\left(\sum_{i=2}^{n} |c_i| \leq n \cdot \log^{2d} s\right) = 1$$

$$\lim_{s \uparrow \infty} \Pr\left(\frac{1}{s^d}\sum_{i=2}^{n} |c_i| \leq \frac{k_2 s^{d-1} \cdot \log^{2d} s}{s^d}\right) = 1$$

$$\lim_{s \uparrow \infty} \Pr\left(\frac{1}{s^d}\sum_{i=2}^{n} |c_i| = 0\right) = 1$$

Thus the sum of sizes of components in the box $B(s)$ excluding the largest one converges to zero with probability one. By 10.34 we obtain

$$\frac{1}{s^d}|c_1| \to p_\infty(\lambda)$$

Thus largest component $c_1$ is crossing for $B(s)$ and all other components have metric diameter greater than $\Phi_s = \log^2 s$.

The intersection of the infinity component $c_\infty$ and a box split separate $c_\infty$ to smaller components, but with probability tending to one there will be only one large component crossing for the box and possibly a lot of small components.

## 2 Critical Thresholds

Finally we show what is known about critical thresholds for different properties of a random geometric graph RGG$(n, r)$ on the box $[0, 1]^d$ ($n$ is the number of points and $r$ is the radius, i.e. the maximal distance for adjacent points). The degree distribution in RGG follows Bin$(n-1, \pi r^2)$.

It is known that every monotone graph property has a sharp threshold $r_c$, i.e. when $r < r_c - \epsilon$ then the property holds for RGG with probability zero and when $r > r_c + \epsilon$ the property holds for RGG with probability one.

For existence of a giant component it is known that critical threshold $r_c$ is approximately $\frac{\sqrt{\lambda_c}}{\sqrt{n}}$ ($\lambda_c$ is approximately 1.44 from the experiments). For the graph connectivity $r_c = \sqrt{\frac{\log n}{\pi n}}$.

It remains open the exact value of $\lambda_c$ for the existence of a giant component and $r_c$ for planarity.