Local algorithms

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Local graph coloring

Graph coloring
problem : Given graph G with maximal degree Δ
what is minimal number of colors such that vertices connected by edge receive different color?
Observation: $\Delta + 1$ colors suffice.

Local coloring

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In local algorithm each vertex first generate some random numbers, then they communicate with small number of neighbors to obtain its color.

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Our algorithm has following properties

Theorem

Given graph G with maximal degree Δ there exist local algorithm which finds coloring of G with $\Delta + 1$ colors.
Moreover with high probability all vertices need to know random numbers of only additional $O(\log n)$ other vertices.

Algorithm overview

Our algorithm has three phases.

1. Color large fraction of G by Δ^4 colors.
2. Extend coloring from phase 1 to whole G.
3. Reduce coloring from phase 2 to coloring by $\Delta + 1$ colors.

Phase 1

1. Color large fraction of G by Δ^4 colors

Each vertex chooses it's color from Δ^4 colors uniformly randomly.
We color vertex by it's color if none of neighbors uses same color.

Phase 1

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- Dead if it was visited has no unvisited neighbor.

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Our search ends when we run out of live vertices.

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A step of our search starts at live vertex v.

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We associate with i-th step random variable X_i which such that

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By Chernoff bound there exist c such that for $l = c \log n$ such that $P[X_1 + X_2 \ldots + X_l \geq 0] < 1/n^2$
This means that after $c \log n$ steps probability that our algorithm did not finished is $1/n^2$ so by union bound probability that there exist vertex which did not finished within $c \log n$ steps is $1/n$

Phase 2

On each connected component we run algorithm recursively with border of vertices from phase 1 fixed.
From claim we can from example say that each component will be colored with high probability in linear time with respect to number of its vertices.

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In worst case we must explore Δ^{Δ^4} connected components which contributes only by constant factor.

Thanks for attention