Applying Graph Pebbling to Resource Allocation

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Background

- Mathematics of Planet Earth focuses on mathematical solutions to world problems
- An increase in severity and frequency of natural disasters has indicated a need for practical solutions in the aftermath
  - Tornadoes in Oklahoma
  - Hurricanes Sandy and Katrina

Problem Statement

- To identify the most efficient choices in allocating resources within an arbitrary system. The system and the resources can be variable; the research is intended to be of use in a broad range of areas. The efficiency of the various options will be gauged by the total cost in resources of a method or process, with more efficient meaning lower total cost and less efficient meaning higher total cost.

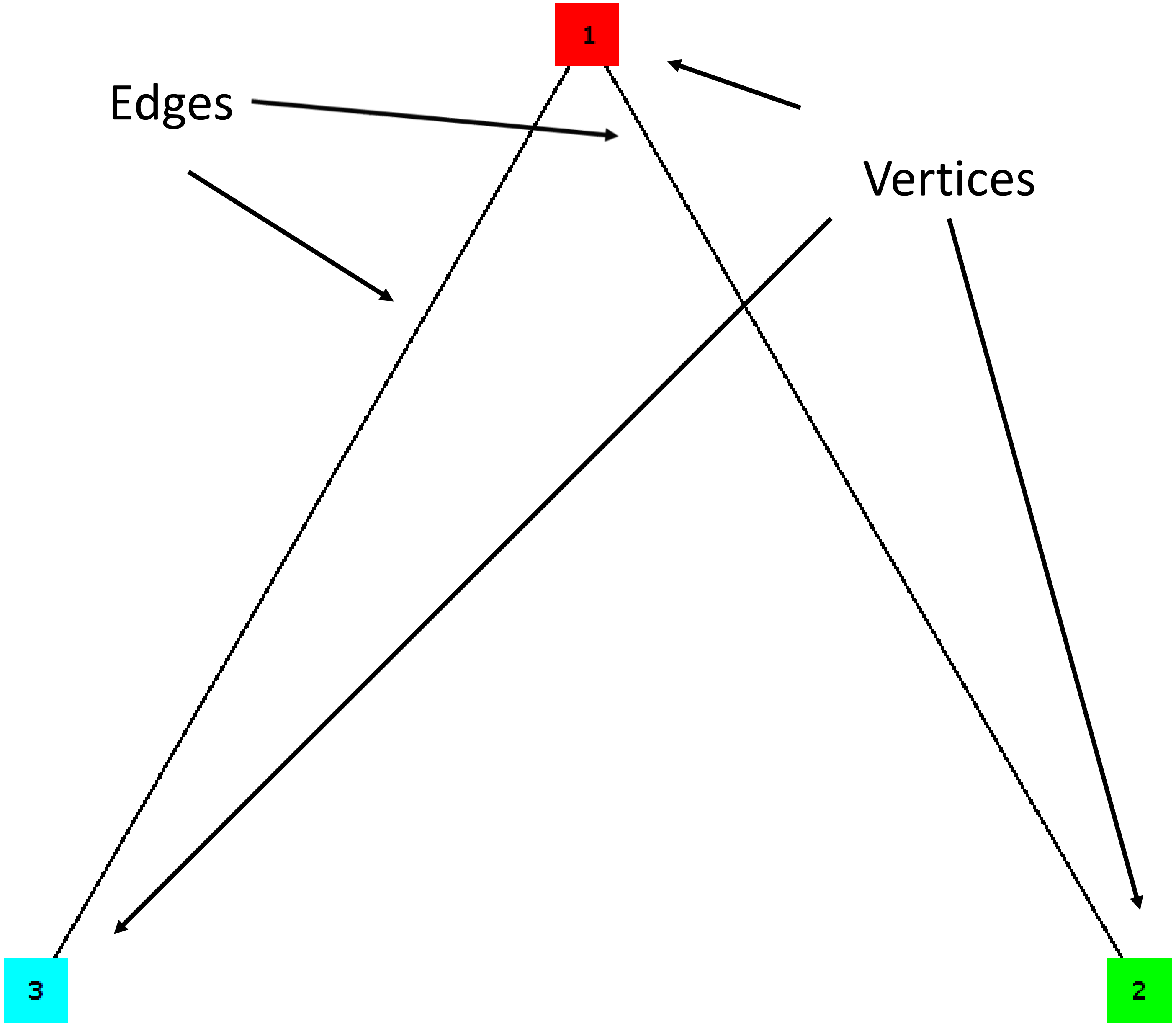
Graph Theory Basics

Graph Theory is the study of Graphs. A **Graph** is an object that consists of:

- A set of vertices, $V(G)$
- A set of edges, $E(G)$
- A function, f, from the set of edges to a set of two vertices.

An **edge** is a connection between two vertices, *u* and *v* ; we say that *u* and *v* are the **endpoints** of the edge. We specify an edge by its endpoints. An edge *uv* is an edge between the vertices *u* and *v*. One can also name an edge *uv* with the set {u,v}. We say that two vertices are **adjacent** if and only if those vertices are connected by an edge.

Visual Representation of Adjacency



Vertices 1 and 3 are adjacent because they are connected by the edge {1, 3}. So are vertices 1 and 2 since they are connected by the edge {1, 2}. However, vertices 3 and 2 are not adjacent; when two vertices are not adjacent, we call them **nonadjacent**.

Graph Types

There are certain Graphs that are commonly used, so they are given special names. A small selection of those graphs is:

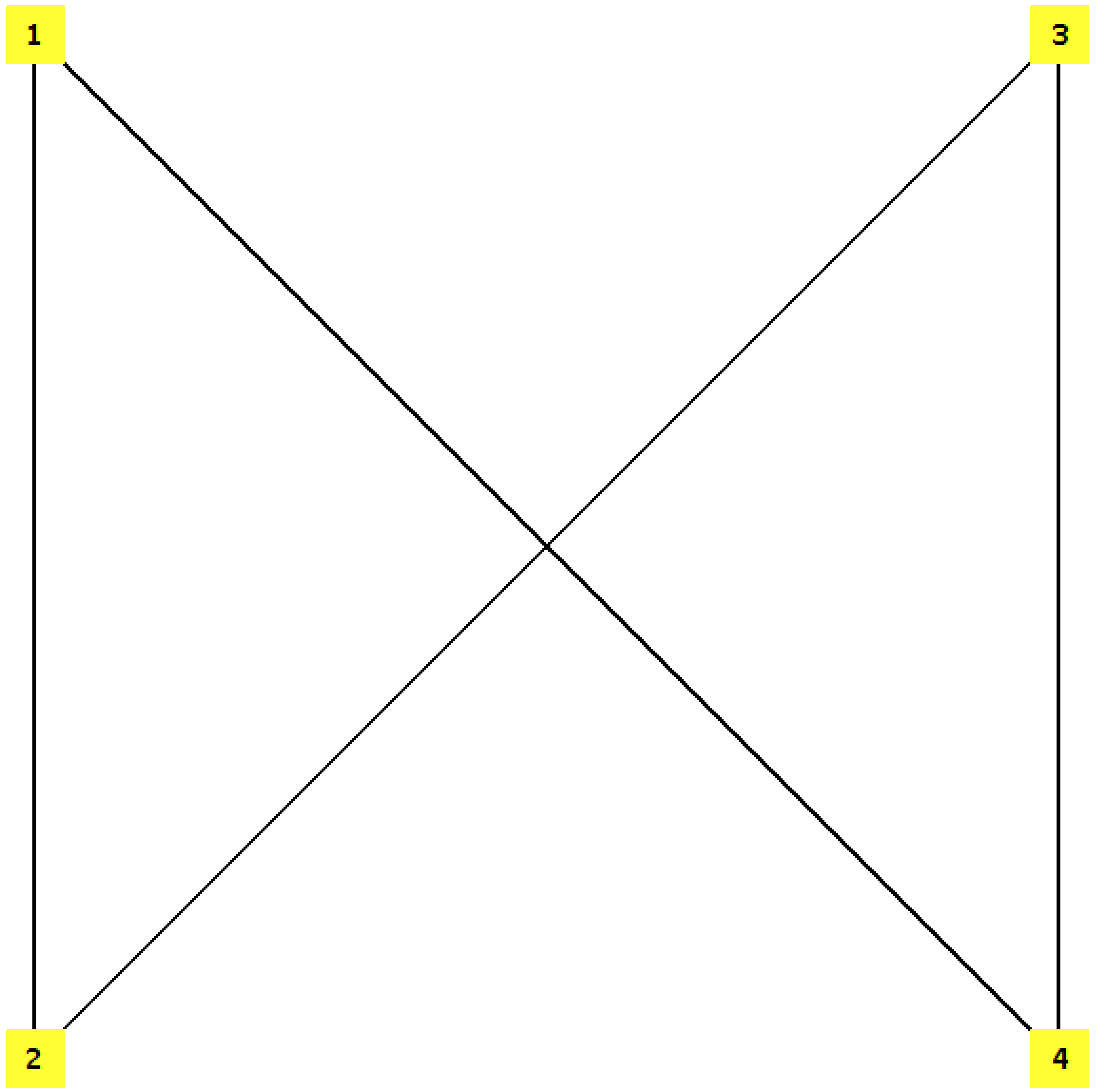
- Path Graphs
- Cycle Graphs
- Complete Graphs

Path Graphs

A path graph is a graph of *n* vertices, with the vertices, v_0 and v_{n-1} called the endpoints of the path. The set of edges in the graph is defined to be $\{\{v_0, v_1\}, \{v_1, v_2\} \ldots \{v_{n-2}, v_{n-1}\}\}$.

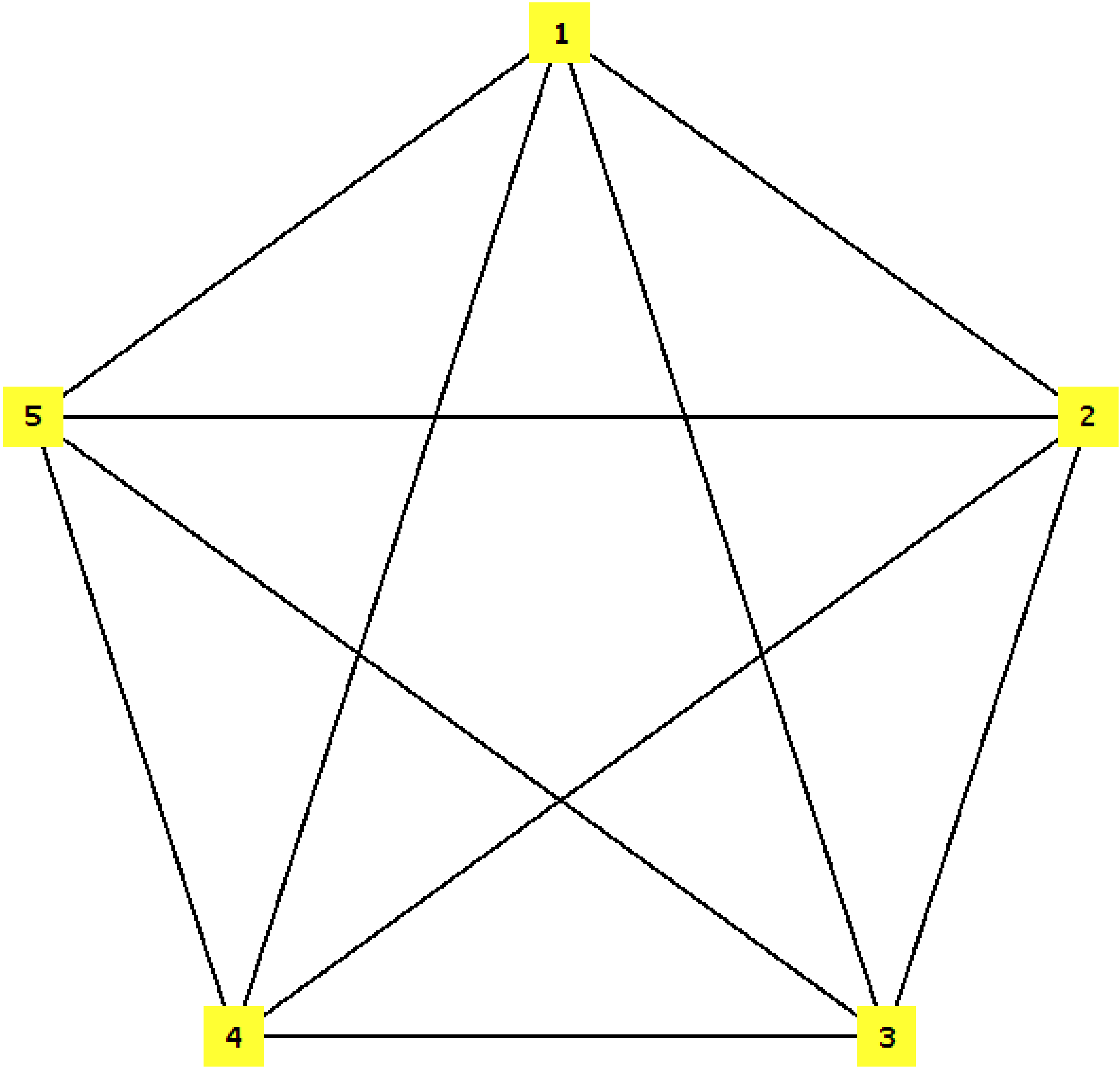
Cycle Graphs

A Cycle Graph is defined as a path graph with an added edge $\{v_{n-1}, v_0\}$.



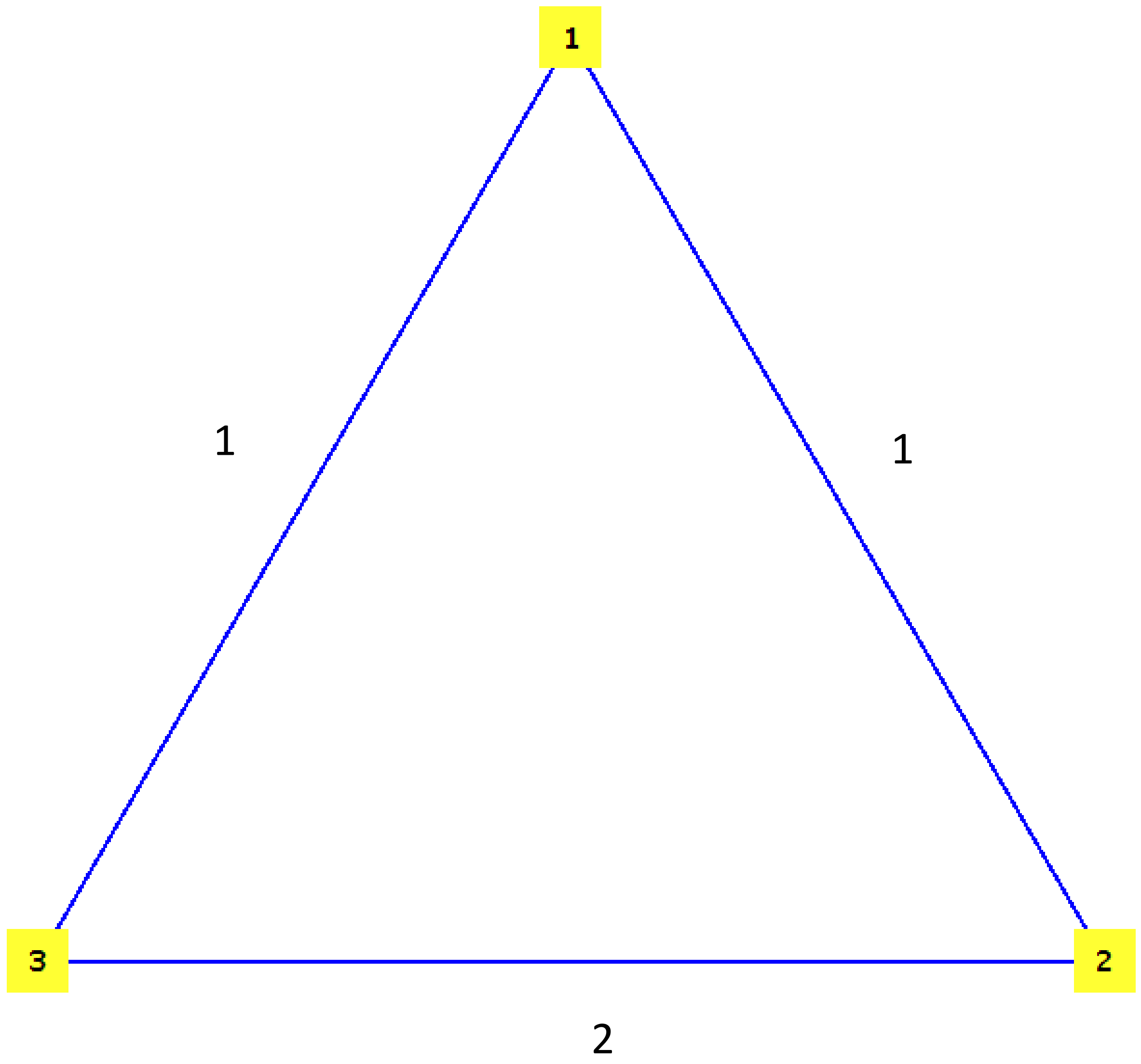
Complete Graphs

A complete graph , K_n is a graph with n vertices where all vertices in the graph are pairwise adjacent. Another way to say this is that for all possible pairs, *{u,v}*, where *u* and *v* are in $V(K_n)$, there is an edge {*u*,*v*}



Weighted Graphs

Weighted Graphs have all of the characteristics of Graphs. Where they differ is that weighted graphs include another function, which relates edges to an integer, *w*, called a **weight**. These weights can be used to model concepts were some edges need to be made distinct from other edges. Here, {1,3} and {1,2} have weight 1 and {3,2} has weight 2.



What is Graph Pebbling?

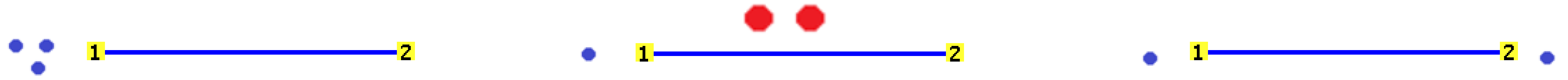
- Concept comes from foundations of Graph Theory, Combinatorial Number Theory and Group Theory.

- Introduced by Dan Kleitman, Paul Lemke and Fan Chung (1989)

Graph Pebbling (Cont.)

Pebbling Move:

Moving pebbles from one vertex to another by removing two pebbles from one vertex and putting one pebble to an adjacent vertex.

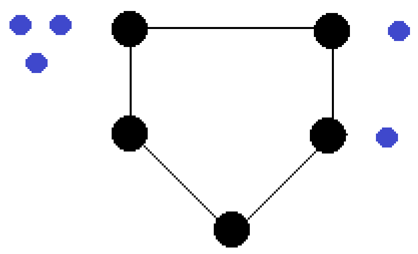


Graph Pebbling (Cont.)

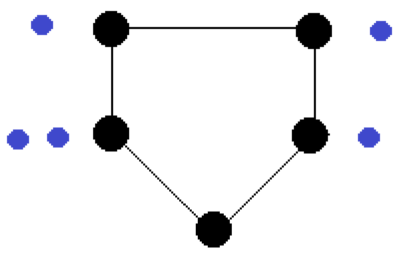
Configuration:

A certain distribution of pebbles on $V(G)$. A configuration is called solvable if every vertex is reachable, otherwise it is unsolvable.

Ex.



Not solvable



Solvable

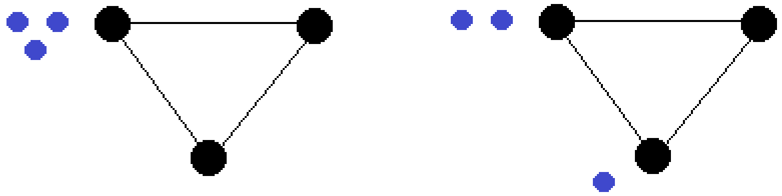
Graph Pebbling (Cont.)

Pebbling Number:

Denoted by $\Pi(G)$

Minimum amount of pebbles needed to reach any vertex with any configuration.

Ex. On a three vertex graph $\Pi(G) = 3$



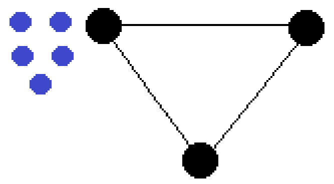
Graph Pebbling (Cont.)

Cover Pebbling: Stacking Theorem

- Denoted by $\gamma(G)$

Minimum amount of pebbles needed to have at least one pebble on every vertex after a series of pebbling moves

Ex. On a complete graph $\gamma(G)$=$2n-1$



Approach

- Use graph pebbling to determine if there is an optimal procedure for moving resources from one location to another where they are needed.
  - Minimum time
  - Minimize use of resources in the transportation process

Approach, continued

- Graph pebbling is going to be used to model the location of the resources, cost of moving them and the paths between locations.
  - Vertices: Destinations
  - Pebbles: Resources
  - Edges: Paths/Routes

Advantages of Approach

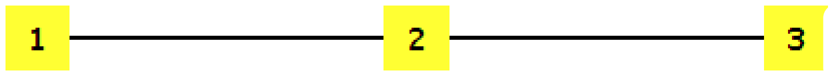
- Analyzes the best route by considering various factors.
- Provides a visual representation of options
- Model is very fluid and adaptable to various situations.

Hypothetical Problem

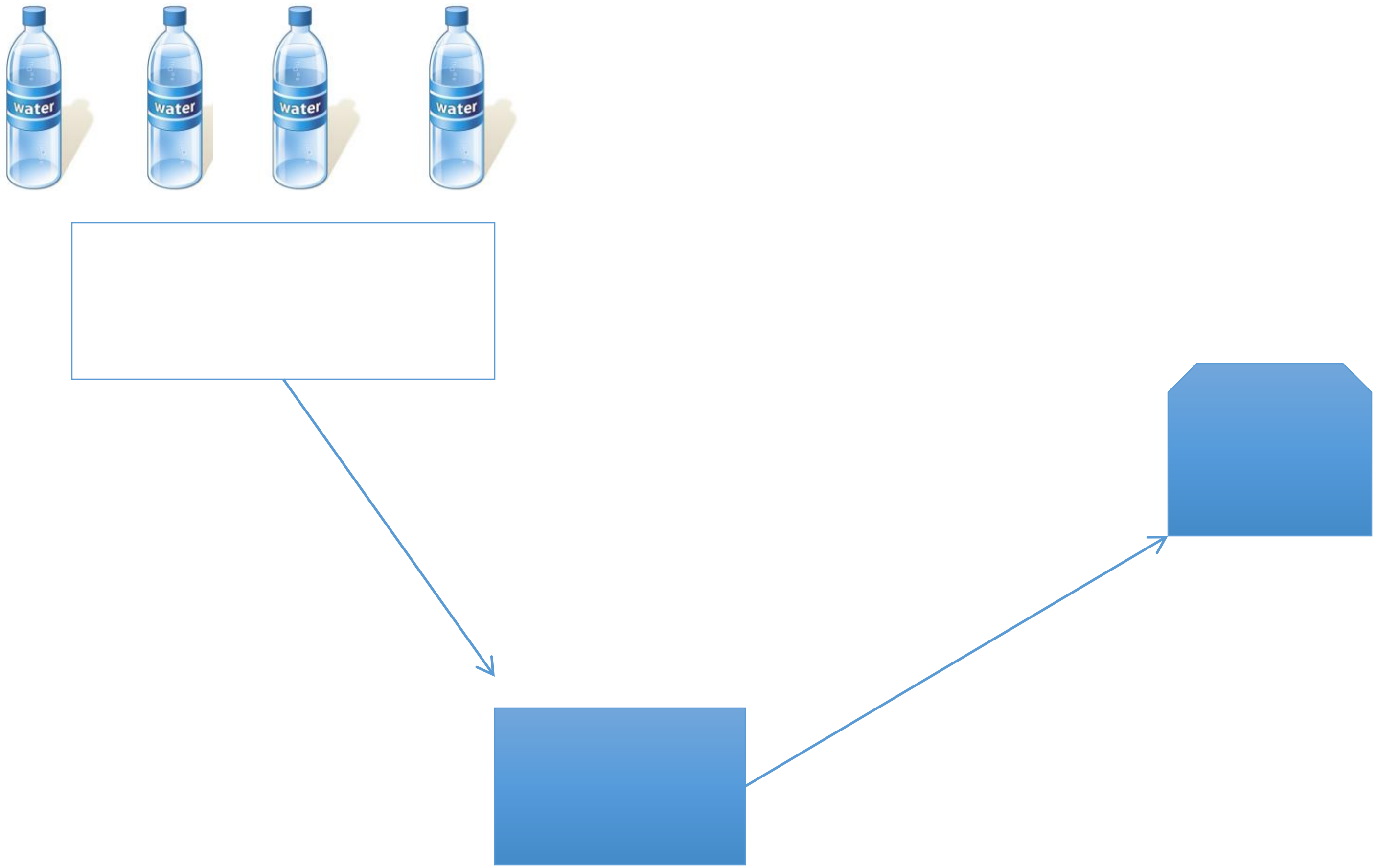
A disaster has struck an area, leaving everyone short of water. A relief group recently came to help the inhabitants of the area, setting up three camps along the main road of the area. Tanks of water can only be delivered to Camp One.

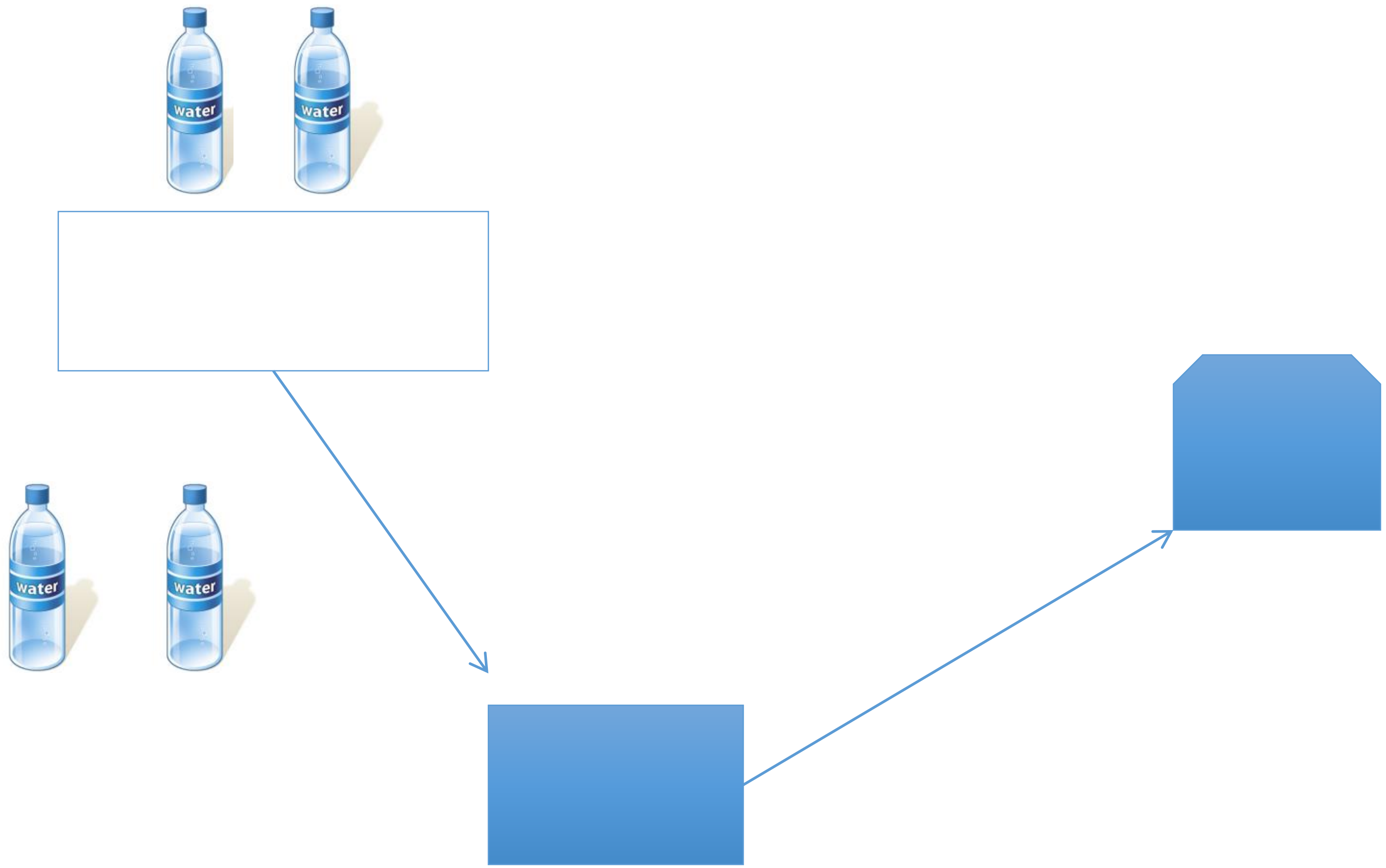
To move tanks of water from camp to camp, trucks must deliver them. Each truck can only hold two tanks per delivery and one of the tanks is consumed by the drivers in order to survive their trip.

1. What is the minimum number of tanks needed to be stored at Camp One so that one tank of water can be successfully delivered to Camp Three?
2. What is the minimum number of tanks needed to be stored at Camp One so that one tank of water can be sent to each camp, and still leave a tank of water for Camp One?



The answer to the first question is four tanks, as these next few slides will show.

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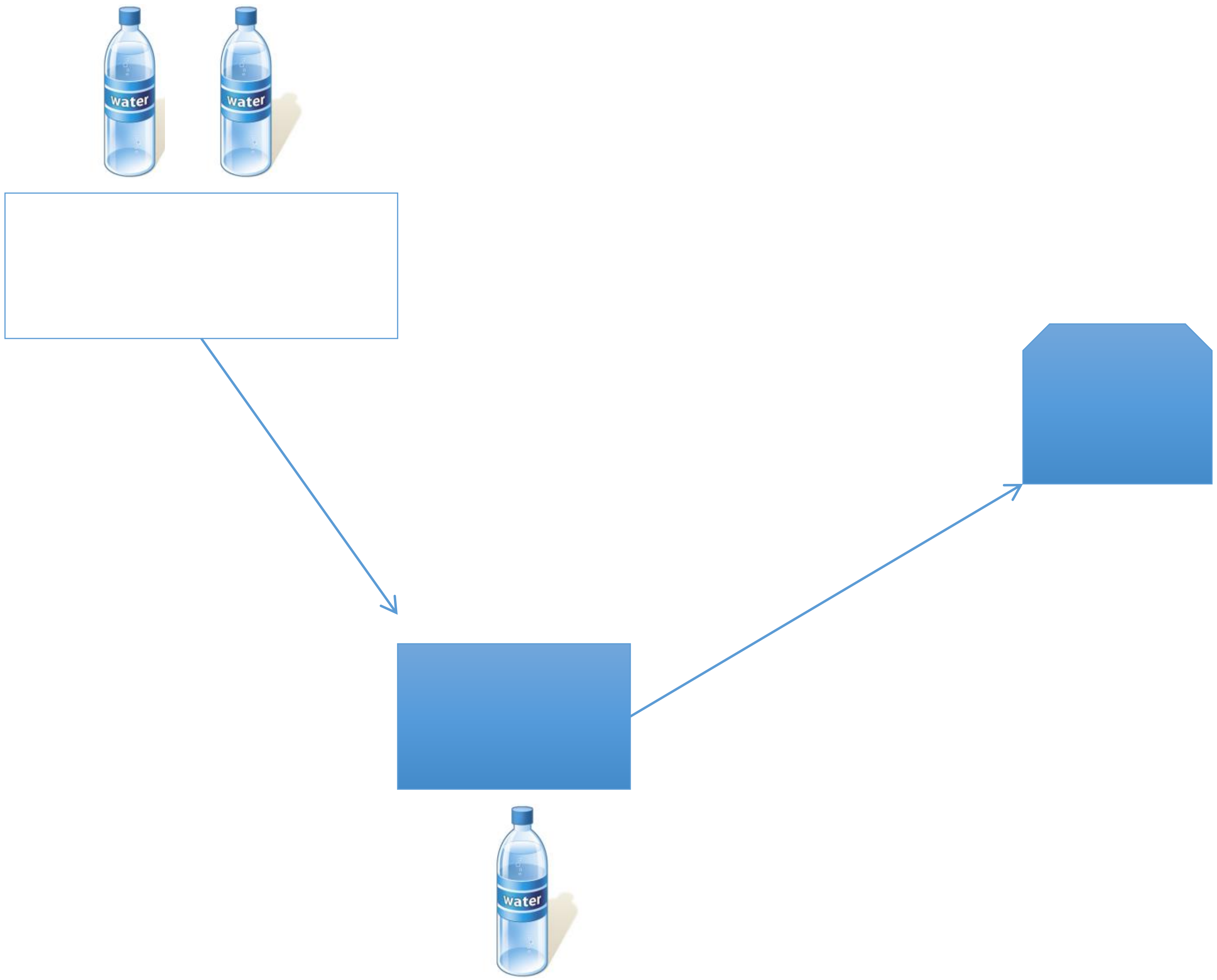
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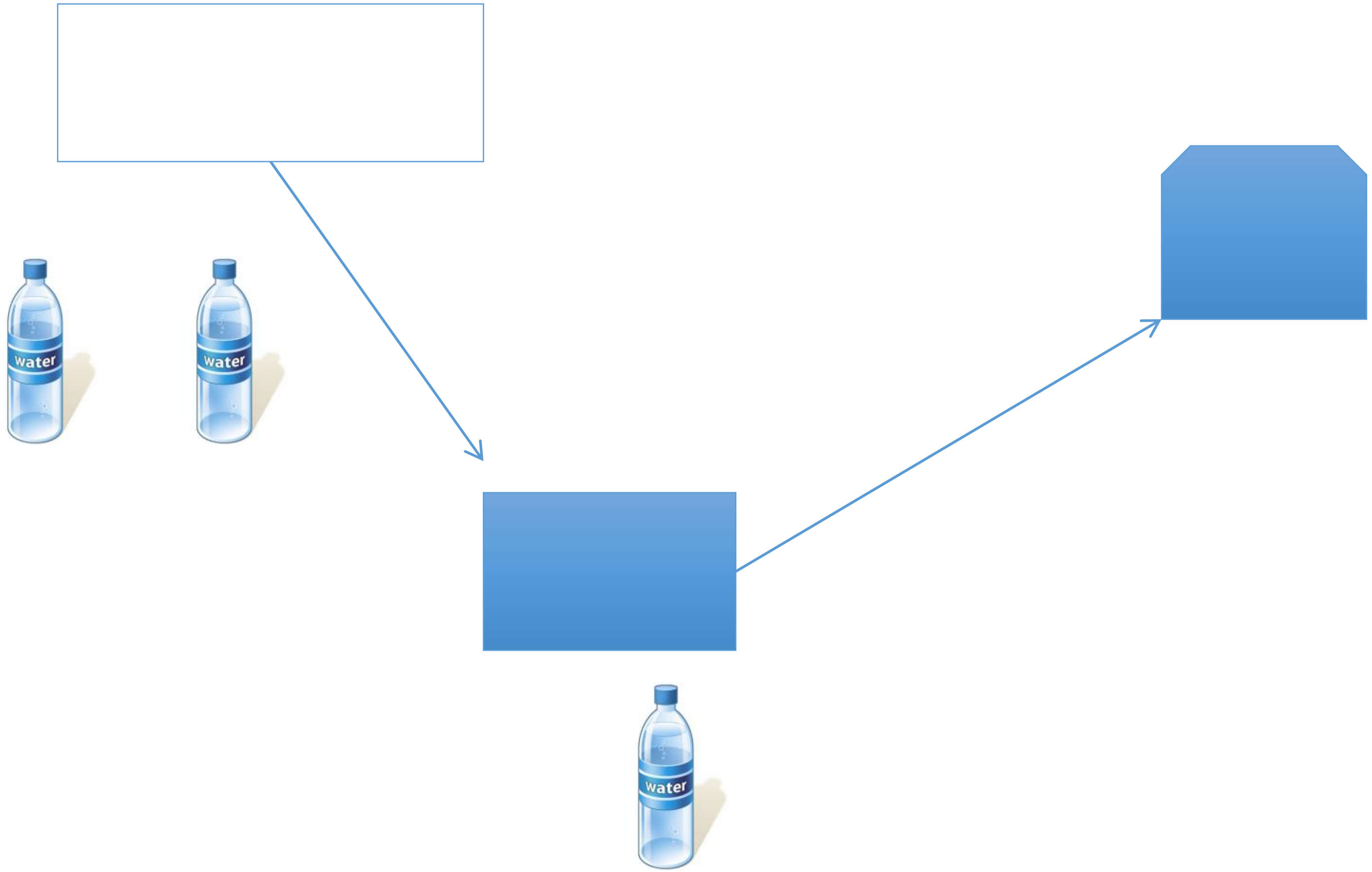
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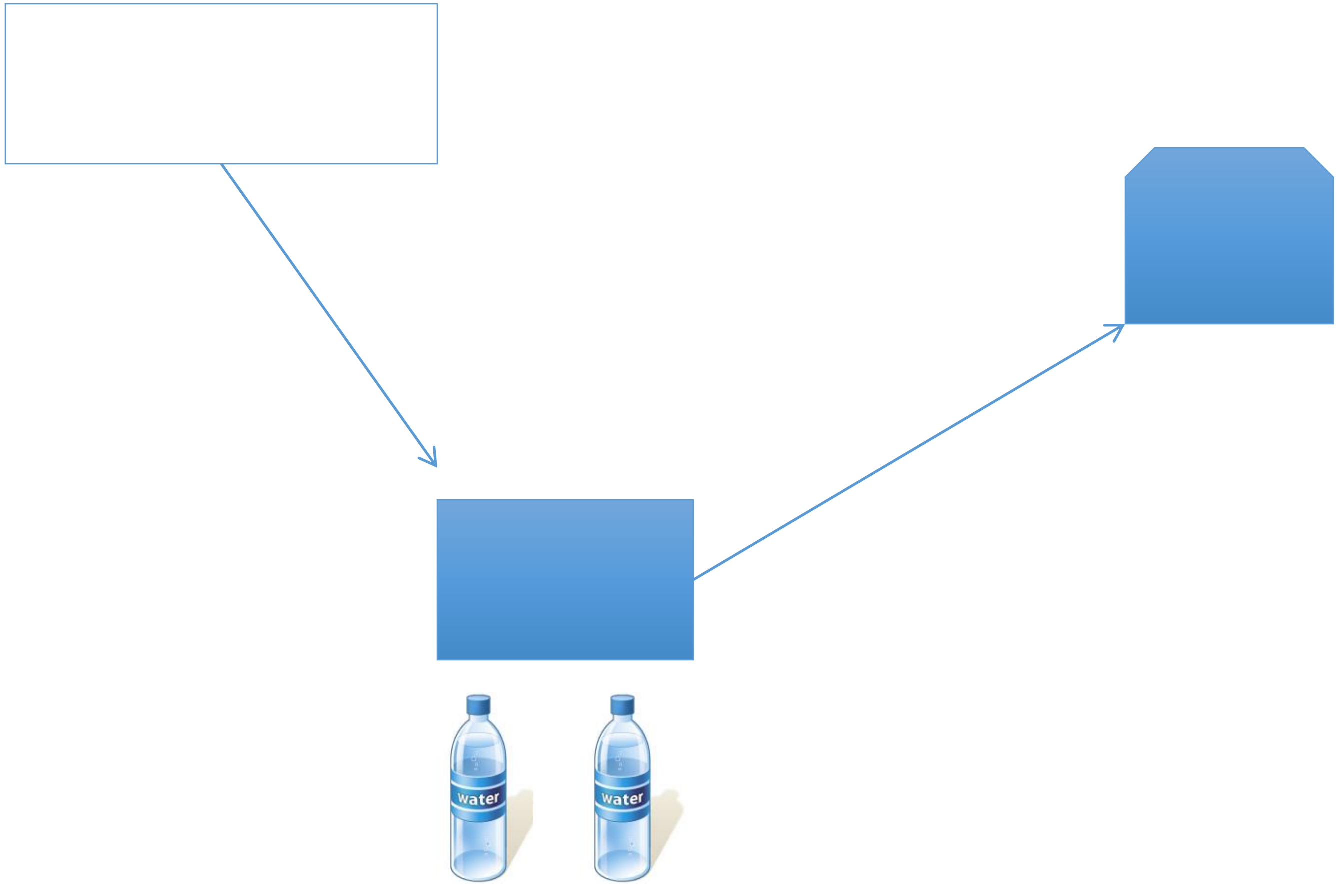
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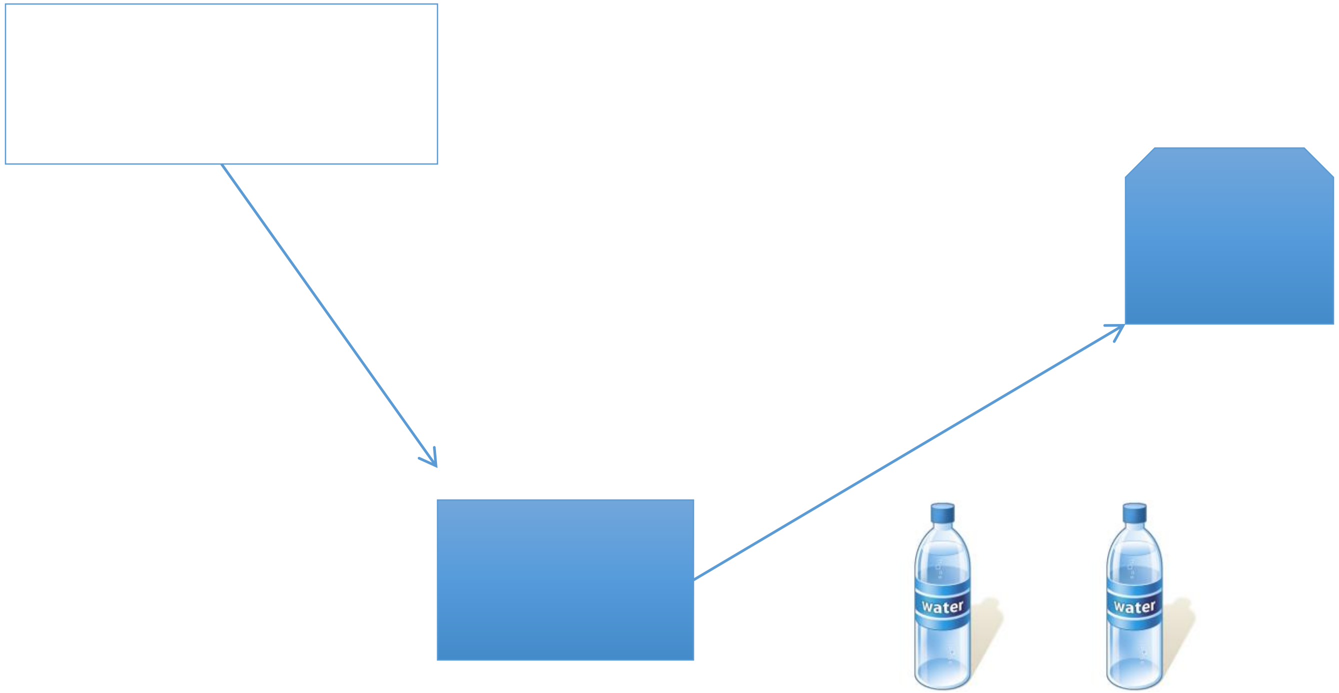
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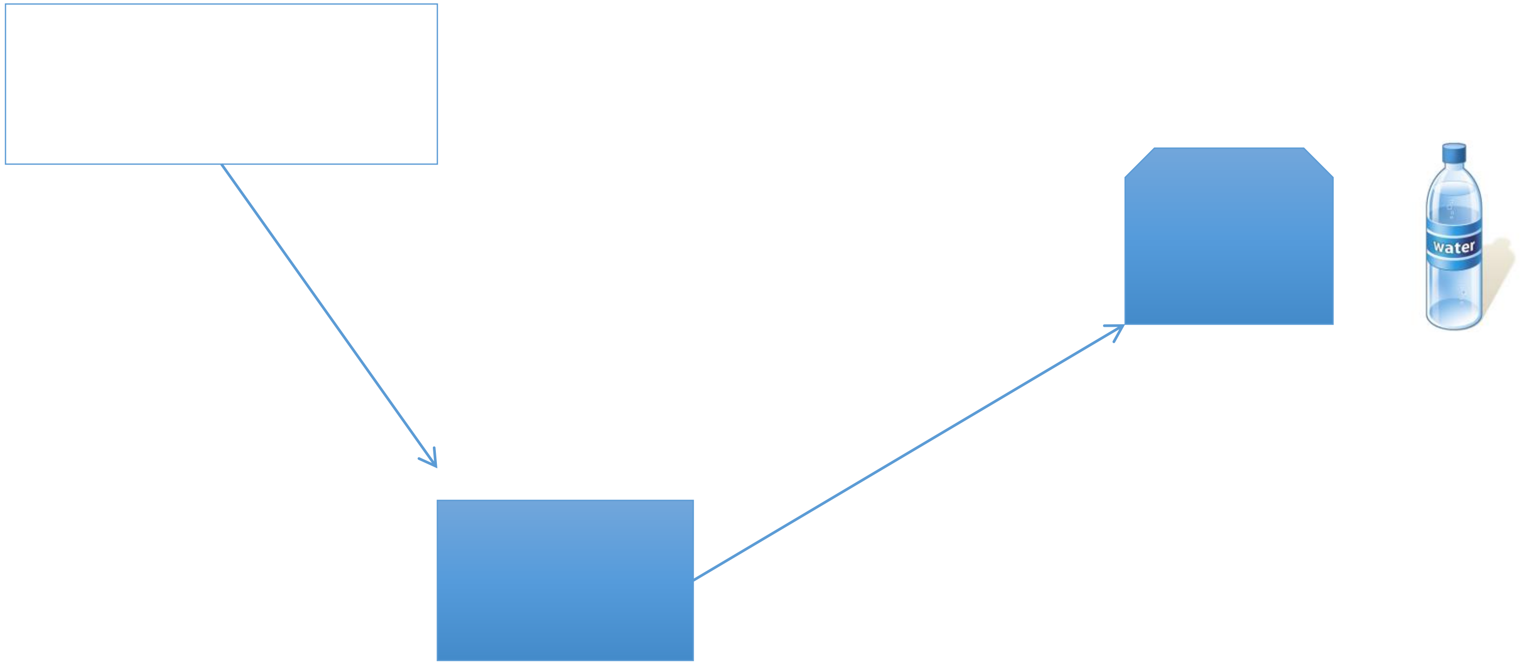
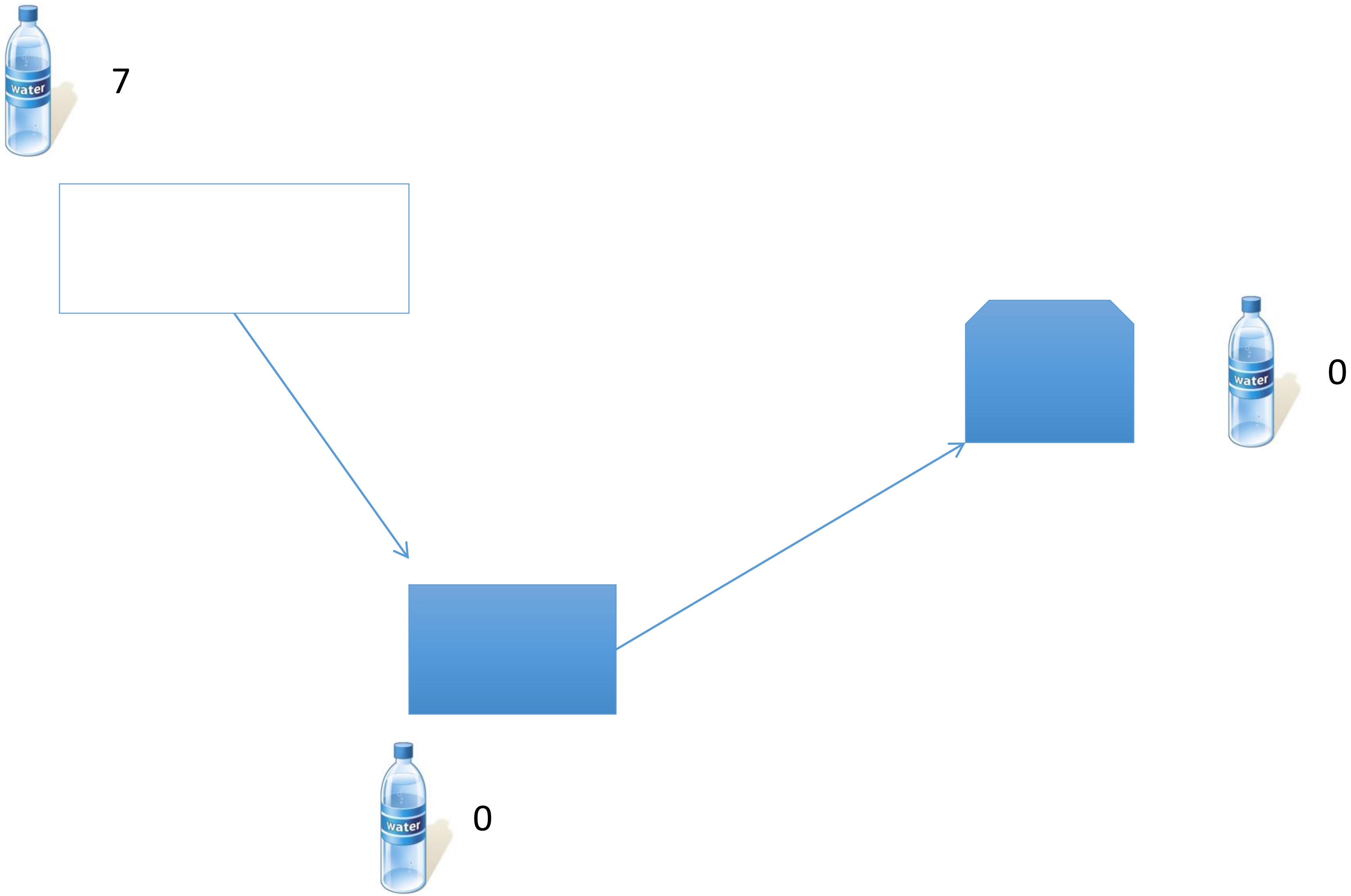
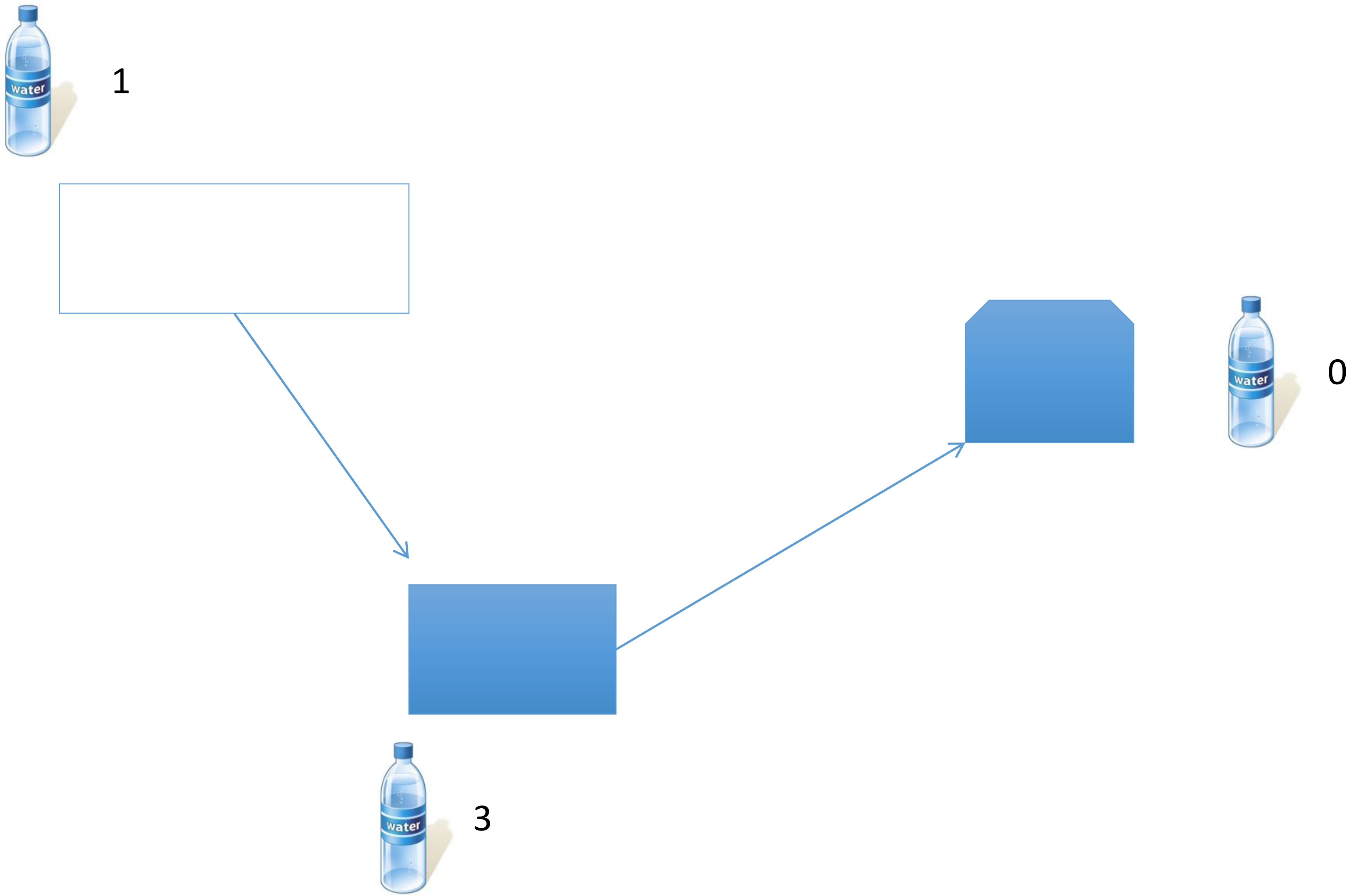
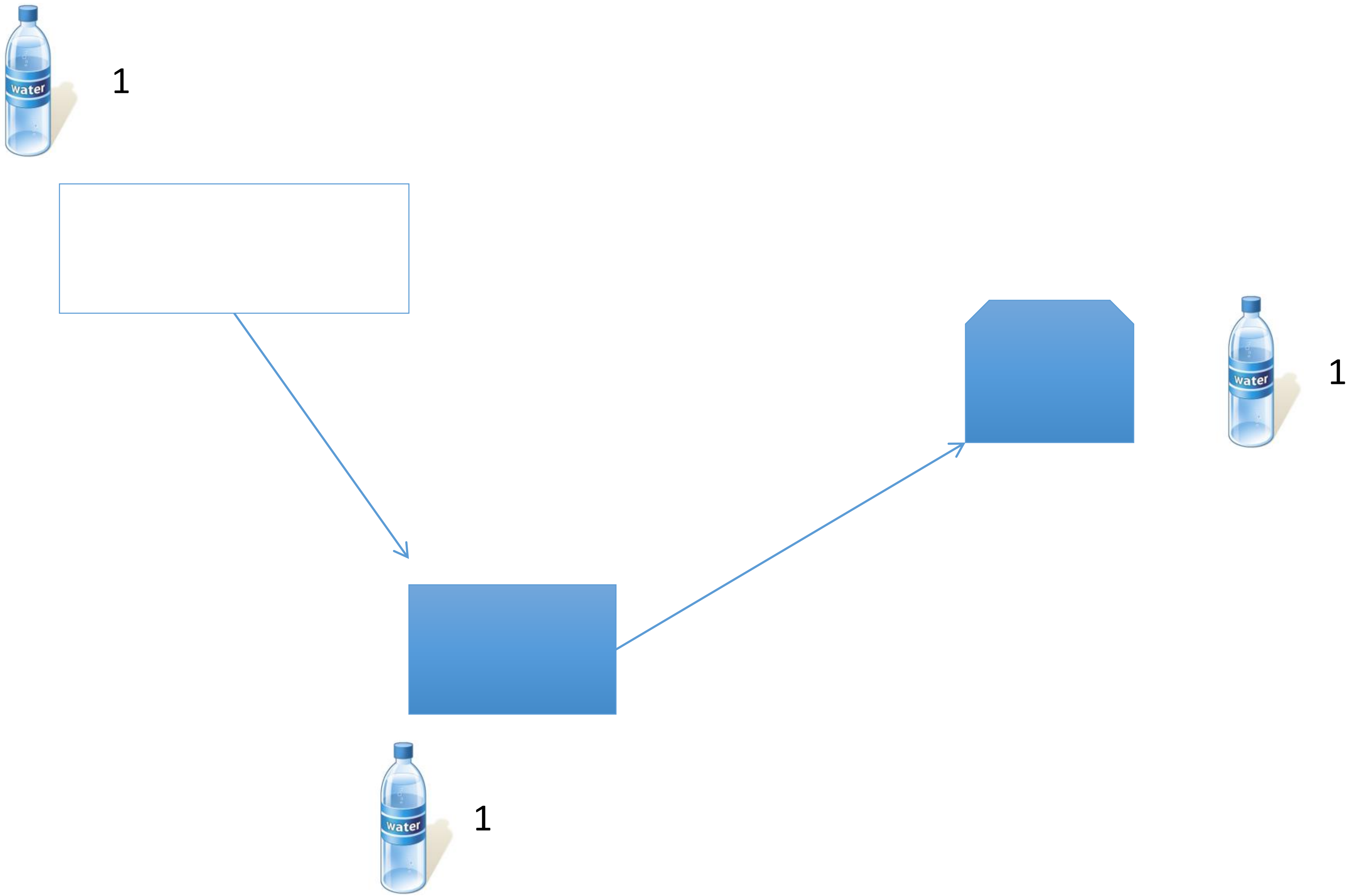

water

The answer to the second question is 7 tanks, as these next few slides will show.




1
0
3


water
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