Construction of Graphs with a Fixed Peeling Number

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Abstract

A peeling number of vertices in a graph are numbers defined by the following process. We repeatedly remove a vertex v with a minimum degree and define a peeling number of v as a maximum degree which we have already removed. Peeling numbers are important for example in social networks. Peeling number tells us "how important vertex is". We focus on graphs where all vertices have the same peeling number. We show how construct these graphs and show that a class of edge extremal graphs equals to generalized trees.

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1 Introduction

We say that a graph G is k-degenerate if every $H \subseteq G$ has a vertex with degree at most k. Specifically we may remove vertices one by one in order such that every removed vertex has degree at most k at the time of removing. This order of vertices we call a k-degenerate-order.

Now we generalize this order such that we always remove a vertex v with a minimum degree in a graph G. Let d be the maximum degree which we have already removed. Then after removing a vertex v we define a peeling number of v as d. It is easy to see that peeling numbers are uniquely determined. We define a peeling-order as an order of vertices with nondecreasing peeling numbers.

We may define a peeling numbers in another way. We have a several of phasions of vertex removing. In a phase i we remove vertices with a degree i or less. Vertex removed during a phase i has a peeling number i. Let us see that both process give us the same peeling numbers.

In the next section we will be interested in graphs where all of their vertices have the same peeling number.

Definition. *Let us denote as FP_k a class of graphs such that $G \in FP_k$ if and only if every vertex $v \in G$ has a peeling number k.*

Observation 1. *Class FP_k contains exactly graphs which are k-degenerate and have a minimum degree equals to k.*

In Section 2, we show a construction of graphs in FP_2, next we define extremal cases of FP_k graphs and find a construction of those graphs. Finally we show that every graph $G \in FP_k$ is a subgraph of some extremal graph from FP_k. Afterwards in Section 3, we show a generalization of trees and show that a class of extremal FP_k graphs equals this class. In Section 4, we show one result for k-degeneracy graphs. Specifically we show that one can find an independent set I in k-degenerate graph G such that the graph $G \setminus I$ is at most $(k-1)$-degenerate.

2 Constructions of FP_k graphs

We would like to find some operations such that we may construct every graph from FP_k using these operations on some set of initial graphs and such that FP_k is closed under these operations. These initial graphs are called primitive graphs with a respect to FP_k and given operations. First, we find operations for the class FP_2 where primitive graphs are isomorphic to disjoint union of triangles and we prove the following theorem.

Theorem 2. *If $G = (V, E)$ is a graph from FP_2 with a component C which is not a triangle, then we can apply one of the following operations to obtain a graph $G' \in FP_2$ where $|V(G)| > |V(G')|$.*

(a) Erase a vertex $v \in V$ such that $\deg(v) = 2$ *and every neighbour u of v satisfies* $\deg(u) > 2$

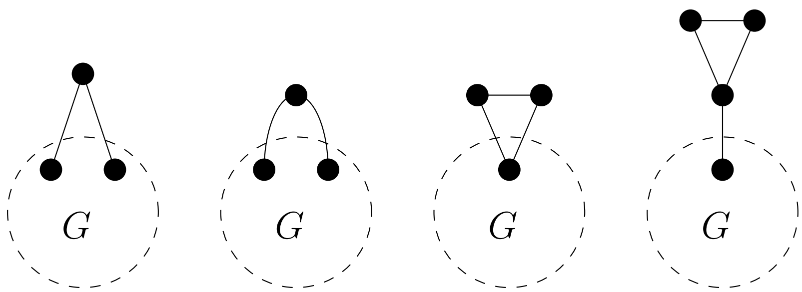

Figure 1: Allowed operations for a construction of graphs from FP_2. From the left to the right there are a vertex adding, an edge subdivision, an edge adding and a triangle adding.

(b) A contraction of an edge $uv \in E$ *such that* $\deg(u) = \deg(v) = 2$ *and* u *and* v *have no common neighbour.*

(c) Erase vertices $u, v \in V$ *such that* $\deg(u) = deg(v) = 2$ *and* u *and* v *have a common neighgbour* w *such that* $deg(w) > 3$.

(d) Erase a triangle uvw *such that* $\deg(u) = \deg(v) = 2 \quad \& \quad \deg(w) = 3$.

Proof. We know that there exists a vertex v with a degree two in every component. We proceed by a case analysis according to the neighbors u_1, u_2 of v where we assume that $\deg(u_1) \leq \deg(u_2)$.

- Both neighbors have degree greater than two. Then we may erase a vertex v. According to 2-degeneracy there still exists another vertex with degree two and according to FP_2 there is no vertex with degree less than two.
- $\deg(u_1) = 2$ and u_2 is not neighbour of u_1. Then we may contract an edge vu_1. The new vertex has degree two.
- $\deg(u_1) = 2 \quad \& \quad \deg(u_2) > 3$ and u_2 is a neighbour of u_1. Then we erase vertices v and u_1 and we still stay in FP_2 because of 2-degeneracy.
- $\deg(u_1) = 2 \quad \& \quad \deg(u_2) = 3$ and u_2 is a neighbour of u_1. If another neighbour of u_2 has degree two, we may contract an edge between it and u_2. In the other case we may erase a triangle vu_1u_2.
- $\deg(u_1) = 2 \quad \& \quad \deg(u_2) = 2$ and u_2 is a neighbour of u_1. So this component is a triangle.

□

Corollary 3. *Every graph* $G \in FP_2$ *can be constructed from triangles by the following operations*

- *adding vertices of degree two,*

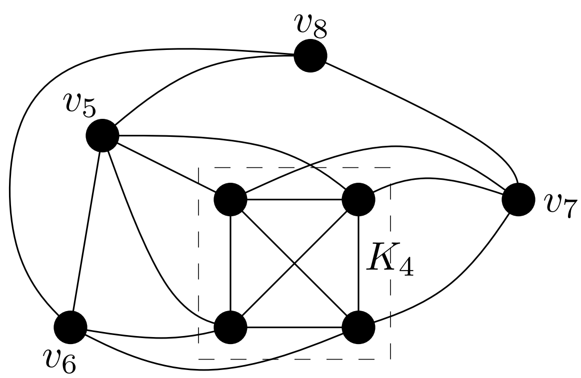

Figure 2: Example of an extremal graph from FP_3. First we have a clique K_3, by adding one vertex we get a clique K_4 and then we add vertices v_5, v_6, v_7, v_8.

- *subdivision of edges,*
- *adding two vertices of degree two connected by an edge,*
- *adding a triangle and connecting it to the rest of the graph by an edge.*

Note that the class FP_2 is closed under these operations.

Now we define extremal graphs in FP_k. We use them later for construction of the class FP_k for other values of k. We say that $G \in FP_k$ on n vertices is extremal if it has the maximal number of edges. Here is the formal definition.

Definition. *We say that a graph $G = (V, E)$ from FP_k on n vertices is extremal if for every edge $e \notin E$ the graph $G + e$ is not in FP_k.*

First we notice that every FP_k extremal graph on n vertices has the same number of edges.

Lemma 4. *Let G be an extremal graph from FP_k on n vertices. Then G has exactly $\binom{k}{2} + k(n-k)$ edges.*

Proof. We know that G is k-degenerate and it has a minimum degree k. So every vertex v_i in k-degenerate-order has $\min\{k, i\}$ forward edges where i is a backward position in that order. If this was not true then we could add a new edge and stay into a FP_k class.

Now it follows that the number of edges is exactly $\binom{k}{2} + k(n-k)$. □

From this fact and k-degenerate-order follows the construction of extremal FP_k graphs.

Claim 5. *Every extremal graph $G \in FP_k$ on $n > k$ vertices can be constructed from a clique K_k by adding $(n-k)$ vertices of degree k.*

Next we prove that any G from FP_k is a subgraph of some extremal graph from FP_k on the same number of vertices. To prove this, we look at k-degenerate-order of G and expand it into k-degenerate-order of some extremal graph.

Theorem 6. *Let G be an graph from FP_k on n vertices. Then there exists an extremal graph $H \in FP_k$ such that H also has n vertices and G is a subgraph of H.*

Proof. We take a k-degenerate-order of G – every vertex $v_i \in G$ has at most $\min\{k, i\}$ forward edges. Additionally we construct an extremal graph H and add to every vertex all forward edges which are also in G. The rest of the edges can be added arbitrarily. $\square$

From the last theorem we get a construction of any graph $G \in FP_k$. First we construct an arbitrary extremal H such that $G \subseteq H$ and then we remove some edges. Let us note that the earlier construction for FP_2 is more convenient for proofs by induction, because we are only adding new vertices and edges and we never delete anything. On the other hand if we want to obtain a result only for extremal graphs from FP_k, then the last construction is also usable.

3 Generalization of trees

In trees every leaf has degree one and the leafs form an independent set and if we remove all of them we obtain another tree. Repeating this process, we end with K_1 or K_2. Now we try to generalize this idea. For a given graph G and the positive integer k we apply the following peeling process.

Algorithm 7. *peel(G, k):*

1. $I := \{v : \quad \deg(v) = k\}$
2. *If I is not independent or there exists a vertex v with degree lower than k then return G.*
3. $G := G \setminus I$
4. *Go to step 1.*

This algorithm may output various graphs. We focus on cases where we end up with a clique K_k or K_{k+1}.

Definition. *Let k be a natural number and define a class of generalized trees $\mathcal{T}_k$ such that graph G is in $\mathcal{T}_k$ if and only if peel$(G, k) = K_k$ or peel$(G, k) = K_{k+1}$.*

We say that a vertex of $G \in \mathcal{T}_k$ with degree k which is not in the final clique K_k or K_{k+1} is a leaf.

The interesting thing, about this class of graphs, is that the class of generalized trees is equivalent to extremal FP_k graphs.

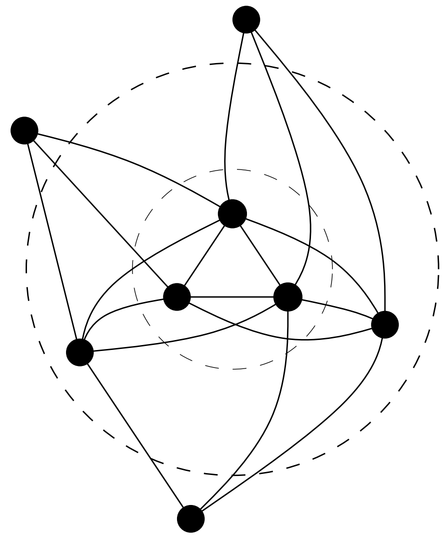

Figure 3: Example of a tree from $\mathcal{T}_3$.

Claim 8. *Let EFP_k be a class of extremal graphs from FP_k. Then for every $k:\ \mathcal{T}_k = EPF_k$.*

Proof. If we take some $G \in \mathcal{T}_k$ then we may repeatively remove some indepentent sets $I_1, \dots, I_l$ of vertices of degree k and end with K_k or K_{k+1}. If we remove the vertices one by one we actually get a construction of extremal case of FP_k. So G is also in EFP_k

For a given graph $G \in EFP_k$ we prove by induction on $|V(G)|$ that G is also contained in $\mathcal{T}_k$. The cliques K_k and K_{k+1} trivially are in $\mathcal{T}_k$. Let us assume that G is a graph with at least $k+2$ vertices. We have a k-degenerate order of G where every vertex except the last k has k forward edges. Therefore every vertex with degree k has only forward edges. So they form an independent set I if we remove this independent set we get $G \setminus I$ which is also from EFP_k and smaller. So $G \setminus I$ is from $\mathcal{T}_k$ and if we add I into this graph as a leafs we get that G is also from $\mathcal{T}_k$. □

Here follows another interesting thing about $\mathcal{T}_k$.

Theorem 9. *For every graph G from $\mathcal{T}_k$ there exist trees $T_1, T_2, \dots, T_k \in \mathcal{T}_1$ such that G is an edge disjoint union of $T_1, T_2, \dots, T_k$ and every leaf in G is also a leaf in every tree T_i.*

Proof. We prove it by induction. For a clique K_{k+1} on vertices $\{1, 2, \dots, k+1\}$, for every $1 \le i \le k$ we define T_i as a graph on vertices $\{1, 2, \dots, k+1\}$ with edges $\{\{i, j\} :\quad j \ge i\}$. Obviously these trees are edge disjoint and every edge is covered.

If we have a larger graph G, then we remove some leaf $l \in V(G)$. By induction on $|V(G)|$ we have trees $T_1, \dots, T_k$ which are edge disjoint and their union is $G \setminus \{l\}$. Now we add a vertex l in all of them such that every T_i stays to be connected. We may do this because there are at most $i-1$ vertices which are not in T_i. So first we choose a neighbor of l which is in T_k, then we choose a neighbor which is in T_{k-1} and so on until the last one which remains is in T_1. Obviously every T_i is still a tree and T_i and T_j are edge disjoint for every i and j. We also covered every edge incident with l. □

4 k-degeneracy and independent sets

Theorem 10. *For every k-degenerate graph $G = (V, E)$ there exists an independent set $I \subseteq V$ such that $G \setminus I$ is at most $(k-1)$-degenerate.*

Proof. We take a k-degenerate order and greedily find a proper coloring of a graph G with colors $\{1, 2, \dots, k+1\}$. We take vertices one by one from the last one to the first one. Everyone of them has at most k forward edges so there exists a color which we may use. We always use the smallest possible color.

Now we claim that a set of vertices with color 1 is the independent set we are looking for. Every vertex v either have color 1 or some of it's forward neighbour have color one. So if a vertex v remains, then its forward degree decrease at least by one and hence the degeneracy of $G \setminus \{v : v \text{ has color } 1\}$ is at most $(k-1)$-degenerate. □

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