Construction of Graphs with a Fixed Peeling Number

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Research Experiences for Undergraduates

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We define FP_k as a class of graphs such that $G \in FP_k$ if and only if every vertex $v \in G$ has a peeling number k.

- Class FP_k contains exactly graphs which are k-degenerate and have a minimum degree equals to k.

Construction

- We want to characterize graphs in the class FP_k.
- We do it constructively.
- We want to find some operations which are closed in FP_k and by which may help us to construct every graph from FP_k.

Construction of FP_2

Theorem

If $G = (V, E)$ is a graph from FP_2 with a component C which is not a triangle, then we can apply one of the following operations to obtain a graph $G' \in FP_2$ where $|V(G)| > |V(G')|$.

(a) *Erase a vertex $v \in V$ such that* $\deg(v) = 2$ *and every neighbor u of v satisfies* $\deg(u) > 2$

(b) *A contraction of an edge $uv \in E$ such that* $\deg(u) = \deg(v) = 2$ *and u and v have no common neighbor.*

(c) *Erase vertices $u, v \in V$ such that* $\deg(u) = deg(v) = 2$ *and u and v have a common neighbour w such that $deg(w) > 3$.*

(d) *Erase a triangle uvw such that* $\deg(u) = \deg(v) = 2$ *and* $\deg(w) = 3$.

Construction of FP_2

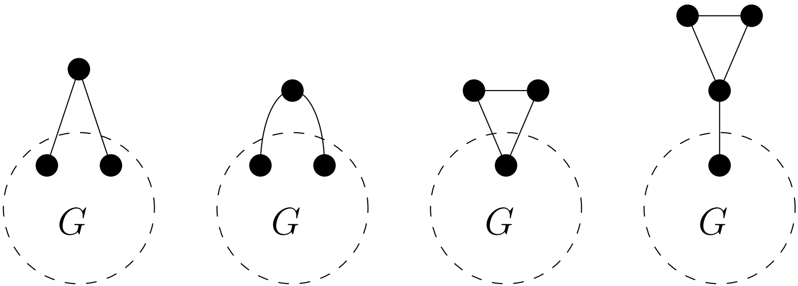


Figure: Allowed operations for the construction of graphs from FP_2. From the left to the right: vertex adding, edge subdivision, edge adding and triangle adding.

Construction of FP_2

Corollary

Every graph $G \in FP_2$ can be constructed from triangles by the following operations

- *adding vertices of degree two,*
- *subdivision of edges,*
- *adding two vertices of degree two connected by an edge,*
- *adding a triangle and connecting it to the rest of the graph by an edge.*

Note that the class FP_2 is closed under these operations.

Extremal cases of FP_k

- It is hard to find a similar construction for $k > 2$.
- First we look only at a construction of an extremal graphs in FP_k.

Definition

We say that a graph $G = (V, E)$ from FP_k on n vertices is *extremal if for every edge $e \notin E$ the graph $G + e$ is not in FP_k.*

- It is useful to look at construction of extremal graphs from FP_k because every graph $G \in FP_k$ is a subgraph of some extremal $H \in FP_k$.

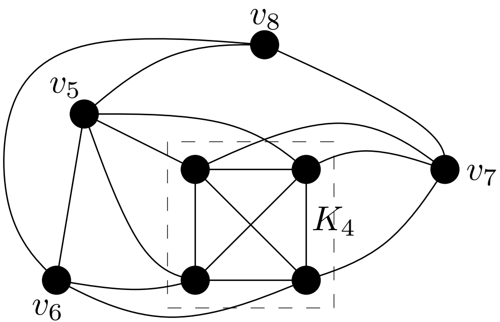
Lemma

Let G be an extremal graph from FP_k on n vertices. Then G has exactly $\binom{k}{2} + k(n-k)$ edges.

Extremal cases of FP_k

Claim

Every extremal graph $G \in FP_k$ on $n > k$ vertices can be constructed from a clique K_k by adding $(n-k)$ vertices of degree k.



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- If we remove all leafs we eventualy get another tree.
- By repeating this process we get K_1 or K_2.
- Now we try to generalize this idea.

Generalization of trees

- For a given graph G and a positive integer k we apply the following peeling process.

Algorithm

peel(G, k)*:*

1. $I := \{v : \quad \deg(v) = k\}$
2. *If I is not independent or there exists a vertex v with degree lower than k then return G.*
3. $G := G \setminus I$
4. *Go to step 1.*

- We focus on cases where we end up with a clique K_k or K_{k+1}.

Generalization of trees

Definition

Let k be a positive integer and define a class of generalized trees $\mathcal{T}_k$ such that graph G is in $\mathcal{T}_k$ if and only if $peel(G, k) = K_k$ *or* $peel(G, k) = K_{k+1}$.

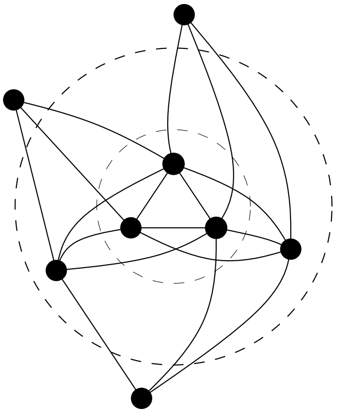

Figure: Example of a tree from $\mathcal{T}_3$.

Generalization of trees

Claim

Let EFP_k be a class of extremal graphs from FP_k. Then for every $k:\ \mathcal{T}_k = EPF_k$.

Theorem

For every graph G from $\mathcal{T}_k$ there exist trees $T_1, T_2, \ldots, T_k \in \mathcal{T}_1$ such that G is an edge disjoint union of $T_1, T_2, \ldots, T_k$ and every leaf in G is also a leaf in every tree T_i.

k-degeneracy and independent sets

- Last remark concerns k-degenerate graphs and independent sets.

Theorem

For every k-degenerate graph $G = (V, E)$ there exists an independent set $I \subseteq V$ such that $G \setminus I$ is at most $(k-1)$-degenerate.

References

- J. Abello and F. Queyroi. "Fixed Points of Graph Peeling," (2013), Advances in Social Networks.
- V. Batagelj and M. Zaveršnik. "An $\mathcal{O}(m)$ algorithm for cores decomposition of networks," (2003), arXiv:cs/0310049.
- D. R. Lick, and A. T. White. "k-degenerate graphs," Canad. J. Math, vol. 22, pp. 1082–1096, 1970.
- A. Montresor, F. D. Pellegrini, and D. Miorandi. "Distributed k-core decomposition," CoRR, vol. abs/1103.5320, 2011.
- S. Seidman, "Network structure and minimum degree," Social networks, vol. 5, no. 3, pp. 269–287, 1983.