The Hardness Boundary for Neighborhood Diversity

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Overview

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  - Courcelle's Theorem (MSO model checking in $O(f(k)n)$)
  - Generally: we look for an algorithm running in $O(f(k)n^c)$, demonstrating that the problem is *Fixed Parameter Tractable* (FPT).

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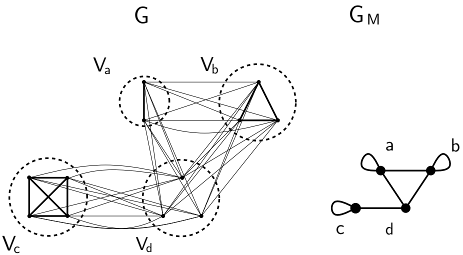
- Equivalence classes are *bags* or *types*
- The bags are either cliques or independent sets; between them are either no edges or all possible edges (a complete bipartite graph)
- Thus we can form a *type graph*:



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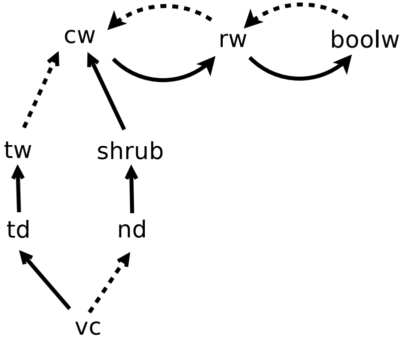
- ND's relationships to other parameters:



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  - CHANNEL ASSIGNMENT (every instance can be reduced to a clique)

The state of the art

Problem	**Treewidth**	**Neighborhood diversity**
PRECOLORING EXTENSION	W[1]-hard	FPT [Ganian12]
L(P,Q)-LABELING	NP-c for TW ≥ 2	FPT [FialaGKK13]
LIST COLORING	W[1]-hard	W[1]-hard
CHANNEL ASSIGNMENT	NP-c for TW ≥ 3	NP-h

The state of the art (contd.)

Problem	Treewidth	Neighborhood diversity
EQUITABLE COLORING	W[1]-hard	FPT [*]
ACHROMATIC NUMBER	NP-c on trees	FPT [*]
CDS	W[1]-hard	FPT [*]
CVC	W[1]-hard	FPT [*]
p-VERTEX-DISJOINT PATHS	W[1]-hard	FPT [Ganian12]
p-EDGE-DISJOINT PATHS	NP-c for TW ≥ 2	Open
Locally constrained homomorphism	W[1]-hard	Open
EDGE BIPARTIZATION	FPT	Open

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  - We could exploit even stronger results than Lenstra's: every Convex Integer Programming problem is FPT w.r.t. the dimension (the number of variables). [Khachiyan, Porkolab '00]

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- What about *shrub-depth*?

Conclusion

Thank you for your attention!