Experiments of the Geometric Modulus Principle

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July 19, 2013

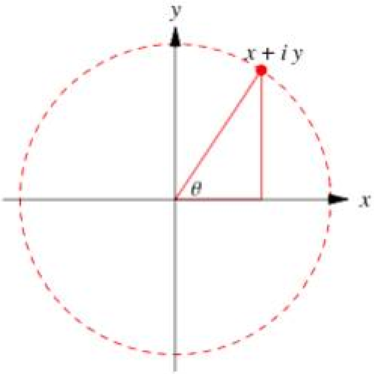
Rough Outline

- Quick Refresher on Terms
- Geometric Modulus Principle
- Experiments and Visualization

Introduction (What is a complex number?)

Definition

A *complexnumber* can be defined as a pair of real numbers, $\mathbb{C}$={(x,y):x,y$\in\mathbb{R}$} where $i = \sqrt{-1}$



Basic Complex Analysis Terms

Definition

The *modulus* gives the length of a complex number from the origin

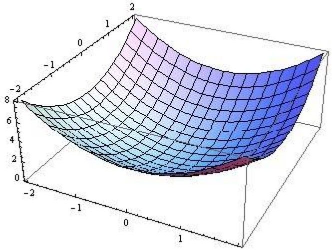
Consider z=x+iy, the modulus of z is:
abs(z)=abs(x+iy)=$(x^2+y^2)^{1/2}$

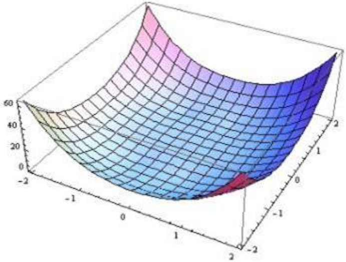
Definition

Let a_0,....$a_n \in \mathbb{C}$. Then for all x$\in \mathbb{C}$, the expression : $\forall x \in \mathbb{C}$:
$P(x)=\sum^{n}{}_{j=0}\ (a_j z_j)=a_0+a_1 z+......+a_{n\text{-}1} z^{n\text{-}1}+a_n z^n$ defines a complex function from $\mathbb{C}$ to $\mathbb{C}$.

We can write $p(z)$ as $p(x, y) = u(x, y) + iv(x, y)$

Modulus Graph

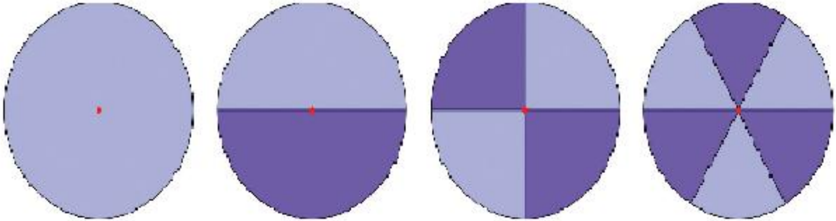




Geometric Modulus

Theorem

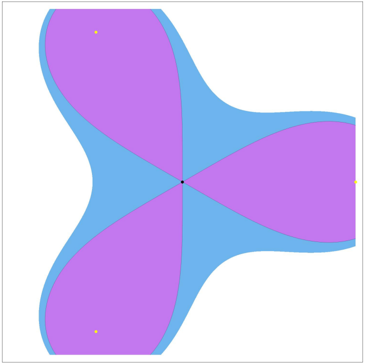
(Geometric Modulus Principle for Polynomials) Let $p(z)$ be a nonconstant polynomial. If $p(z_0)=0$, then every direction at z_0 is an ascent direction for abs($p(z_0)$). If $p(z_0)\neq 0$, thenthe cones of ascent and descent direction at z_0 partition the unit disc centered at z_0 into alternating ascent and descent sectors of equal angle π/k where $k\geq 1$ is the smallest index with $p^k(z_0)\neq 0$



Geometric Modulus cont.

Example: Given a function p(z)=z^3-1. We start by taking a few derivatives and evaluate them at z=0.

- $p'(z)=3z^2$
- $p''(z)=6z$
- $p^3(z)=6$



Theorem

(Hermite Interpolation for Polynomials) Assume $f \in C[a,b]$ *and* $x_0,\ldots,x_n \in [a,b]$ *are distinct points. Then the unique polynomial of degree less than or equal to* $2n+1$ *is given by:*
$H_{2n+1}(x)=\sum^{n}{}_{j=0} f(x_j)H_{n,j}(x)+\sum^{n}{}_{j=0} f'(x_j)\hat{H}_{n,j}(x)$ *where*
$H_{n,j}(x)=[1-2(x-x_j)L'_{n,j}(x_j)]L^{2}{}_{n,j}(x_j)$ *and* $\hat{H}_{n,j}(x)=(x-x_j)L^{2}{}_{n,j}(x_j)$

Construct a polynomial which has specific properties of ascent and descent

Experiment cont.

z	p(z)	p'(z)
2	0	1
-2	0	1
2i	0	1
2i	0	1
1+i	1	0
1-i	1	0
-1+i	1	0
-1-i	1	0

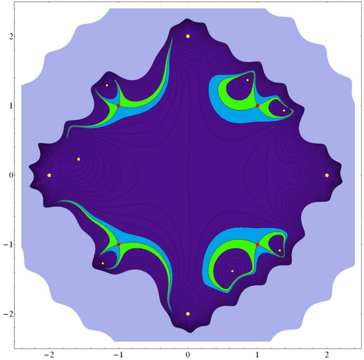


Figure : Batman

- Take the modulus of a complex polynomial p(z)
- Take a point $z_0 = x_0 + iy_0$
- Evaluate $f_0(x_0,y_0) = abs(p(z_0))$, then in a box containing z_0 for many points evaluate f(x,y). Whenever it is larger than $f_0(x_0,y_0)$ color it red and if it is smaller than $f_0(x_0,y_0)$ color it black.

Visulation cont.

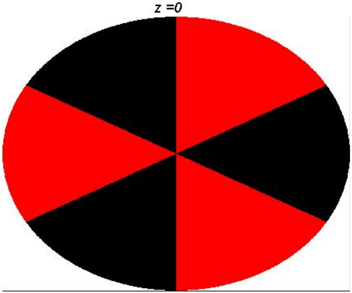


Figure : p(z)$=$z^3-1

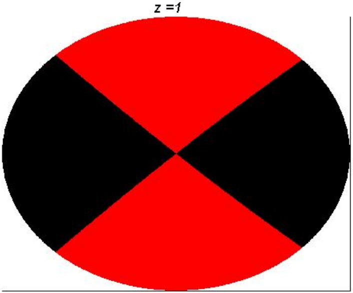
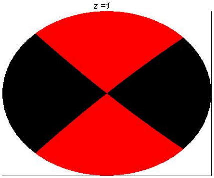
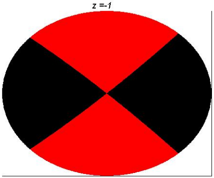


Figure : p(z)$=$z$^4\frac{1}{4}$-z

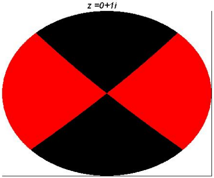
Visulation cont.2



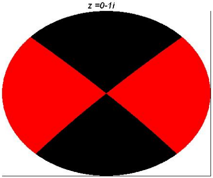
(a) z=1



(b) z=-1



(c) z=i



(d) z=-i

Figure : $p(z)=z^{5}\frac{1}{5}\text{-}z$

Further Work

- Extend Hermite Intepoloation to Higher Derivatives
- Global Gradient

- Bahman Kalantari, A Geometric Modulus Principle for Polynomials, The American Mathematical Monthly, Vol. 118, No. 10 (December 2011) (pp. 931-935).
- B. Kalantari and B. Torrence, The Fundamental Theorem of Algebra for Artists: Math Horizons, April 2013.

Thanks :

- DIMACS REU Program
- Bahman Kalantari
- Yan Wang