Experiments with Geometric Modulus Principle for Polynomials

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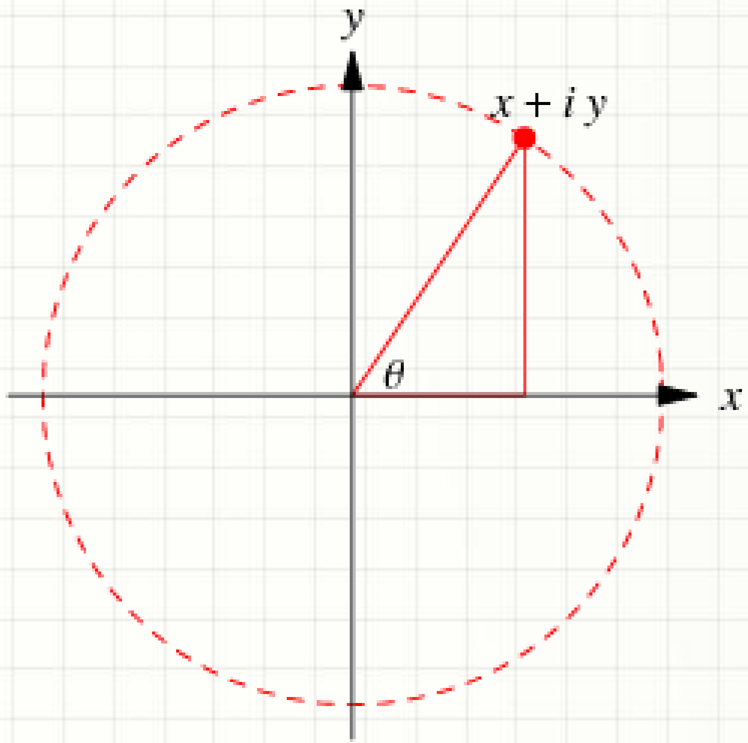
Complex Numbers

- The complex numbers are the field $\mathbb{C}$ of numbers of the form a+bi where a,b$\in$ $\mathbb{R}$ and $i = \sqrt{-1}$
- $a = R_e z$ (real part)
- $b = I_m$z (imaginary part)
- Also denoted as (a,b)



Fundamental Theorem Of Algebra

- The **fundamental theorem of algebra** states that every non-constant complex polynomial has at least one complex root.
- A complex polynomial is the expression of this form $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ where $a_n \neq 0$
- Many proofs for FTA
- One proof uses The Liouville Thereom: A bounded entire function is constant.

Visualizing a Complex Polynomial

- Domain and range are each two- dimensional
- Four dimensions are required
- One visualization uses the modulus

Modulus

- Gives the norm of Complex Number
- Distance from origin
- Consider z=x+iy, the modulus of z is

$$|z| = |x + iy| = \sqrt{x^2 + y^2}$$

Where x and y are real components

- We can write p(z) as $p(z) = u(x, y) + v(x, y)i$

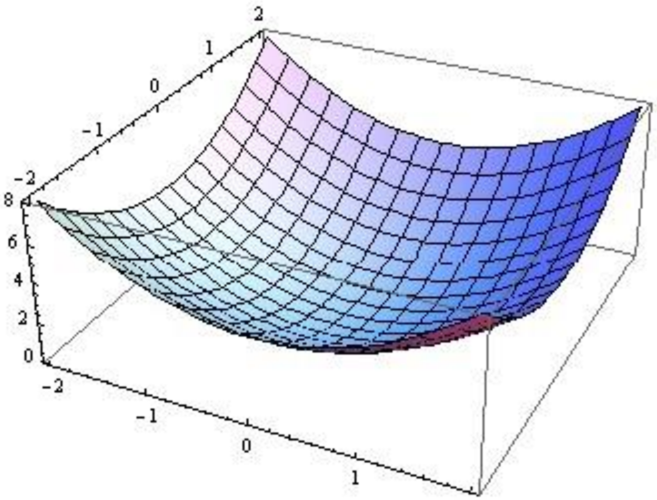
in which u (x,y) and v(x,y) are real functions of two variables.

Modulus cont.

- For example if $p(z) = z^2 - 1$ where $z = x + iy$, we have $p(z) = (x + iy)^2 - 1 = (x^2 - y^2 - 1) + 2ixy$
- Note that $u(x, y) = x^2 - y^2 - 1$ and $\mathrm{v(x, y)} = 2xy$
- Hence $|p(z)| = \sqrt{(x^2 - y^2 - 1)^2 + 4x^2y^2}$

Modulus Graph

- To construct a modulus graph of a complex function p
- Use the x-y plane to represent the domain
- Use vertical axis to represents $|p(x+iy)|$
- FTA implies that the modulus graph of any polynomial must dip down an touch the x-y plane
- Can look at the 2D contour plot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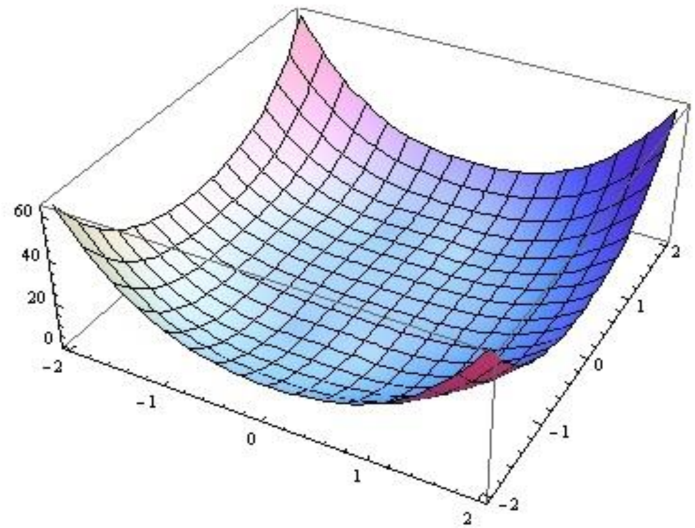
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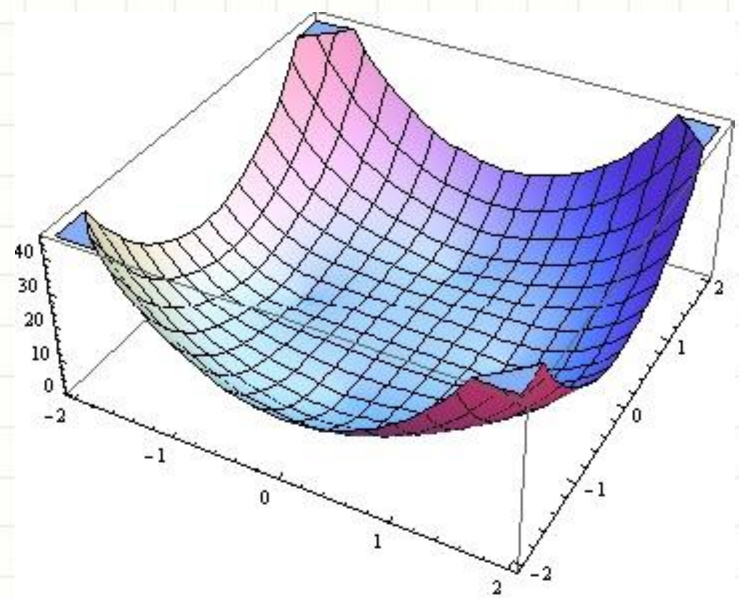
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Properties of Modulus

- **Geometric Modulus Principle Theorem**: Let $p(z)$ be a nonconstant polynomial. If $p(z_0) = 0$, then every direction at z_0 is an ascent direction for $|p(z)|$. If $p(z_0) \neq 0$ then the cones of ascent and descent direction at z_0 partition the unit disc centered at z_0 into alternating ascent and descent sectors of equal angle ${}^{\pi}/_{k}$ where $\mathrm{k} \geq 1$is the smallest index with $p^k(z_0) \neq 0$. (Bahman Kalantari 2011)
- Used to prove Maximum Modulus Principle, FTA, Guass-Lucas Theorem

Geometric Modulus Principle cont.

- Given a function $p(z) = z^3 - 1$ a disk centered at the origin would be divided into six ascent/descent sectors
- We start by taking a few derivatives and evaluate them at z=0

$$p'(z) = 3z^2, p''(z) = 6z, p^3(z) = 6$$

- Observe that z=0 is a root of $p'$and p''
- The third derivative at z=0 does not vanish
- The GMP implies central angle ${}^{\pi}/_{3}$

Graph of $p(z) = z^3 - 1$

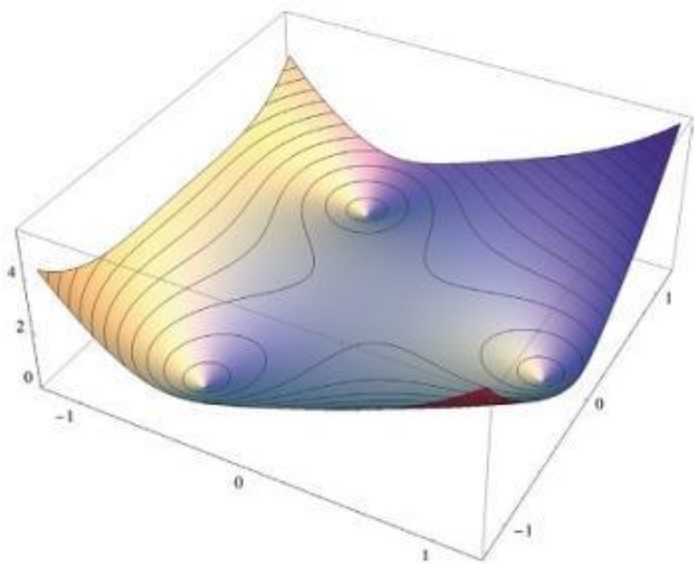
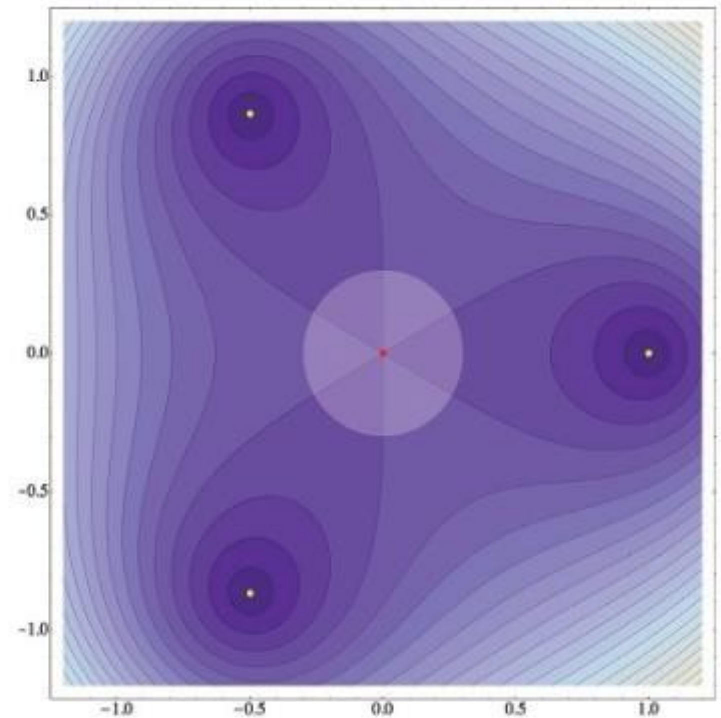


Figure 1. The 3D modulus graph of $p(z) = z^3 - 1$.



Problem/Approach

- The significance of the GMP
- Construct polynomial will give me a specific properties of ascent and descent
- Polynomial Interpolation
- Hermite Interpolation

References

1. Bahman Kalantari, A Geometric Modulus Principle for Polynomials, The American Mathematical Monthly, Vol. 118, No. 10 (December 2011) (pp. 931-935)
2. B. Kalantari and B. Torrence, The Fundamental Theorem of Algebra for Artists: Math Horizons, April 2013