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The Triangle Algorithm : A Geometric Approach to Systems of Linear Equations

Introduction

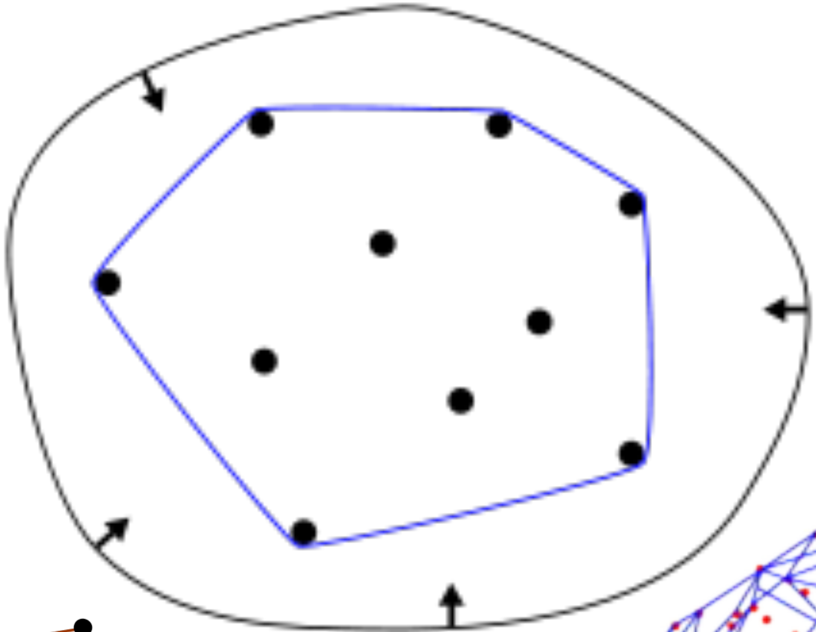
- **Definition:**

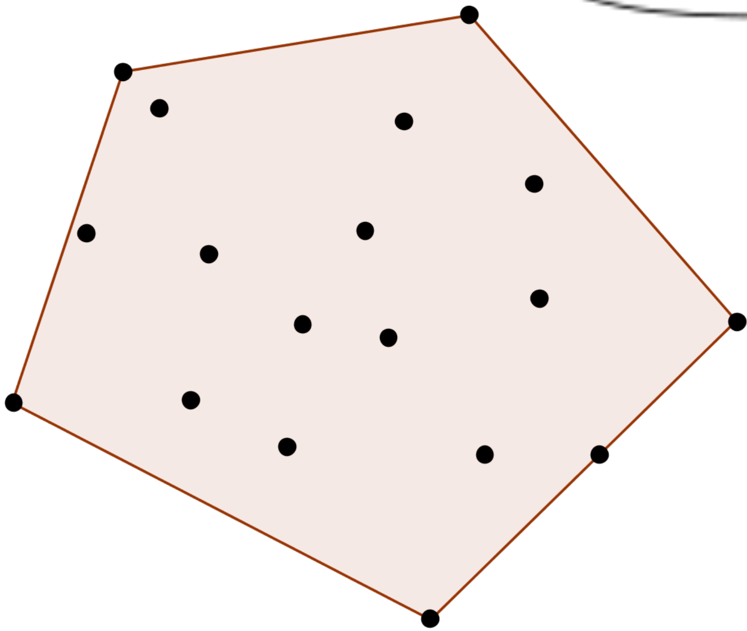
Given a finite set of points S in $\mathbf{R}^m$, the convex hull is the minimal convex set containing S.

The convex hull of a finite point set S is the set of all convex combinations of its points. Written as:

$$conv(S) = \left\{ \sum_{i=1}^{|S|} \alpha_i x_i \,\middle|\, (\forall i : \alpha_i \geq 0) \wedge \sum_{i=1}^{|S|} \alpha_i = 1 \right\}.$$

The convex hull of a finite point set in $\mathbf{R}^m$ forms a **convex polygon**





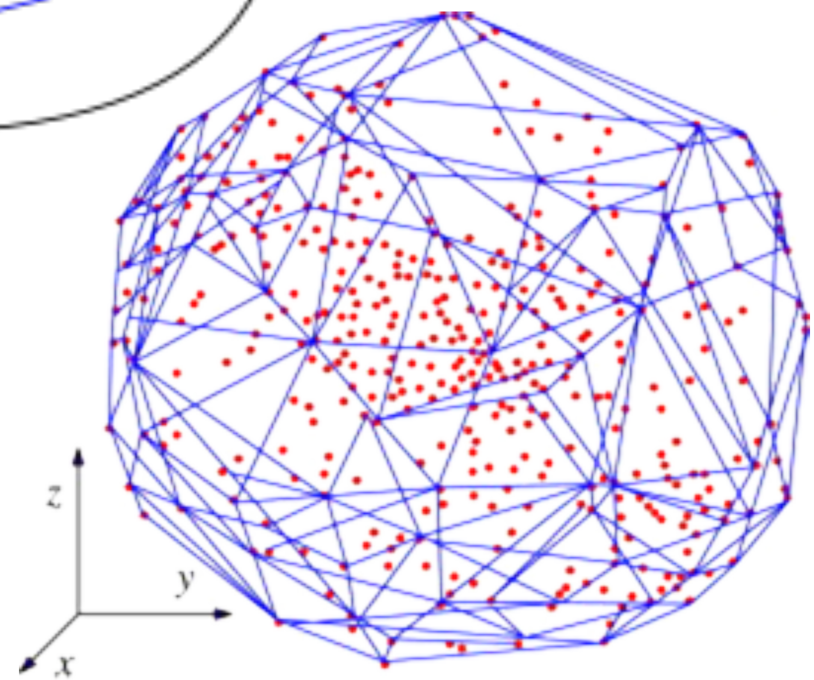


Image source(Clock-wise from top): http://en.wikipedia.org/wiki/Convex_hull, http://www.sciencedirect.com/science/article/pii/S0734743X12001467, http://hcmop.wordpress.com/tag/convex/

The Convex Hull Decision Problem

Given a finite set of points in $\mathbf{R}^m$, we must first determine whether or not a distinguished point, p, is in the convex hull.

How do we accomplish this? First, an important theorem:

Distance Duality Theorem: Let $S = \{v_1,\ldots,v_n\} \subset \mathbf{R}^m$, $p \in \mathbf{R}^m$. Let d(u, w) denote the Euclidean distance between the points u and w.

i) $p \in$ *conv(S)* if and only if given any $p' \in$ *conv(S)*, there exists v_j such that $d(p', v_j) \geq d(p, v_j)$.

ii) p is not in the *conv(S)* if and only if there exists $p' \in$ *conv(S)* such that $d(p', v_j) < d(p, v_j)$, $\forall$ i. In this case we call p' a witness. (B. Kalantari 2012)

The Triangle Algorithm

- Given a p', search for a triangle $\triangle pp'v_j$ where $v_j \in S$, $p' \in$ *conv(S)* \ {p}, such that $d(p', v_j) \geq d(p, v_j)$. The point p' is the **iterate**
- If $\triangle pp'v_j$ satisfies the inequality above, v_j becomes a **pivot point** to "pull" the current iterate, along the line segment, closer to p to get a new iterate $p'' \in$ *conv(S)*.

  Note: Since p'' is a convex combination of p' and v_j, it will remain in *conv(S)*.
- If no such triangle exists, then p' is a **witness** certifying that p is not in *conv(S).*

My Project

- Application to systems of linear equations:
  - Think of $Ax = b$ as $A\alpha = p$, where the columns of A are the vertices, alpha is a column of convex coefficients, and p is our distinguished point.
    - Suppose A is nxn invertible, and $x = A^{-1}b \geq 0$. It follows that solving this system is equivalent to approximating $0 \in conv(a_1, a_2, \ldots, a_n, -b)$.
    - What if we don't know if x is nonnegative? Let $e = (1,\ldots,1)^T \in \mathbf{R}^m$. Then there exists a $t \geq 0$ such that if x is a solution to $A(x - te) = b$, then $x = A^{-1}b + A^{-1}(Ate) \geq 0$. We may apply the convex hull approach.

- Application to eigenvalue problems will be considered

Methodology

- Test the Triangle Algorithm against traditional methods of solving systems of linear equations:
- Examples: Jacobi iteration, Gauss-Seidel Method
  - Possibly Krylov Subspace Methods:
    - Arnoldi Iteration, Full Orthogonalization Method (FOM), Conjugate Gradient Method, and Generalized Minimum Residual Method (GMRES).
- Implementation through Matlab
- We will be looking at improving convergence rates and comparing computational efficiency.

Concluding Remarks

- Implementing Dr. Kalantari's Triangle Algorithm presents a unique and exciting new way at looking at solving systems of linear equations
- Applications in linear programming
- Possible applications in Eigenvalue problems may also be considered

Acknowledgements

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References

- B. Kalantari. A Characterization Theorem and an Algorithm for a Convex Hull Problem.
- B. Kalantari. Solving Linear System of Equations Via A Convex Hull Algorithm. 29 Oct, 2012.