

Polynomiography: From the Fundamental Theorem of Algebra to Art

Bahman Kalantari

There is a sense in which an important result in mathematics is never finished.

—Steve Smale [1]

Polynomiography provides a bridge between the well-known Fundamental Theorem of Algebra (FTA) and art. It does so by turning the visualization of polynomial equations into a serious medium for creating artwork of great variety and diversity through a combination of human creativity and computer power. An individual 2D image—a *polynomiograph*—is created based on the use of *iteration functions*: algorithms for solving polynomial equations.

Formally, a polynomial, written as $p(x)$, is defined as a linear combination of integral powers of the variable x . For example, $p(x) = 10x^{48} - 11x^{24} + 1$.

A *root* or *zero* of the polynomial is a value of x for which $p(x)$ equals zero, that is, a *solution* to the polynomial equation $p(x) = 0$.

Throughout the history of science, reaching back to the Sumerians in the third millennium B.C., the task of approximating the zeros of polynomials has been one of the most influential in the development of mathematics. It has been studied by the most famous of mathematicians and even today remains a practical problem in every branch of science.

The *degree* of $p(x)$ —the highest exponent of x —and the *coefficients* of the powers of x describe the polynomial. The FTA is a magical property that always guarantees at least one solution to any polynomial equation of degree at least 1. In fact, there are as many solutions to a polynomial equation as its degree. The solutions need not be distinct. The theorem owes its name to Carl Friedrich Gauss, one of the greatest mathematicians of all time, but its validity was conjectured long before him. Indeed, *complex numbers* were discovered as the result of the solving of quadratic equations such as $x^2 + 1 = 0$, which have no *real number* as a solution.

A complex number is merely a point in the flat *Euclidean plane*, also called the *complex plane*. It is written as the ordered pair (a, b) , indicating its horizontal and vertical coordinates, respectively. Algebraically, this is written as $a + ib$ where i is the square root of minus one ($i^2 + 1 = 0$). Complex numbers inherit the four elementary operations on the real numbers (i.e.

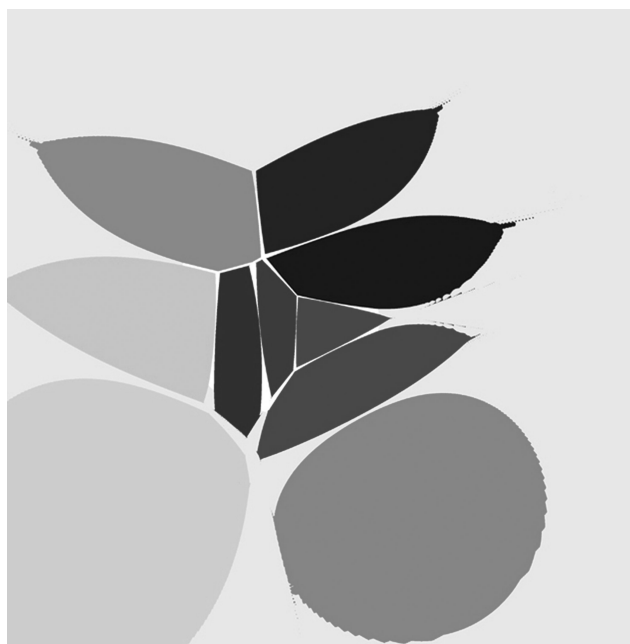
addition, subtraction, multiplication and division). It follows from the FTA that a polynomial equation is an algebraic description of a set of points in the Euclidean plane, namely its roots. Conversely, any set of points in the plane can be written as a polynomial equation having those points and only those points as its solutions.

Polynomiography can be considered *painting via points*, an art form capable of creating an interesting variety of images by manipulating a finite set of points, whether given explicitly, generated by a polynomial equation or selected with the click of a mouse. This view of polynomiography should not be confused with *pointillism*, a term first used with respect to the work of the artist Georges Seurat. In a sense, polynomiography is a minimalist and abstract art form, albeit one of enormous

ABSTRACT

The author introduces *polynomiography*, a bridge between the Fundamental Theorem of Algebra and art. Polynomiography provides a tool for artists to create a 2D image—a *polynomiograph*—based on the computer visualization of a polynomial equation. The image is dependent upon the solutions of a polynomial equation, various interactive coloring schemes driven by *iteration functions* and several other parameters under the control of the polynomiographer's choice and creativity. Polynomiography software can mask all of the underlying mathematics, offering a tool that, although easy to use, affords the polynomiographer infinite artistic capabilities.

Fig. 1. *Summer*, a polynomiograph based on Voronoi coloring of fewer than a dozen points together with personal choice of coloring.
(© Bahman Kalantari)



Bahman Kalantari (educator), Department of Computer Science, Rutgers University, Hill Center, Busch Campus, New Brunswick, NJ 08903, U.S.A.
E-mail: <kalantari@cs.rutgers.edu>

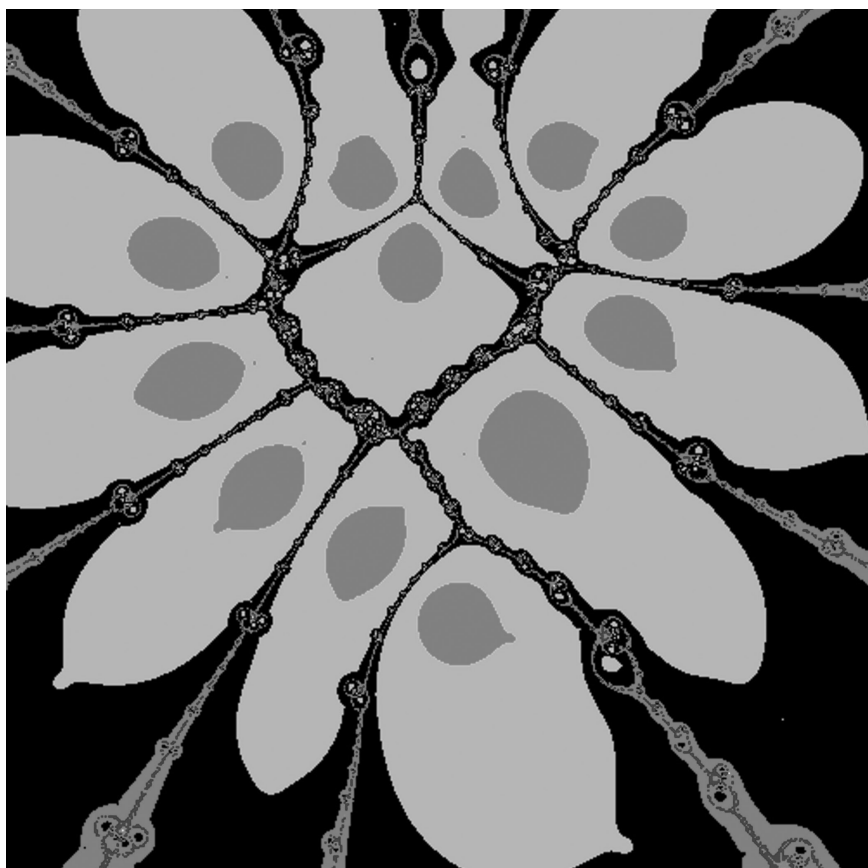
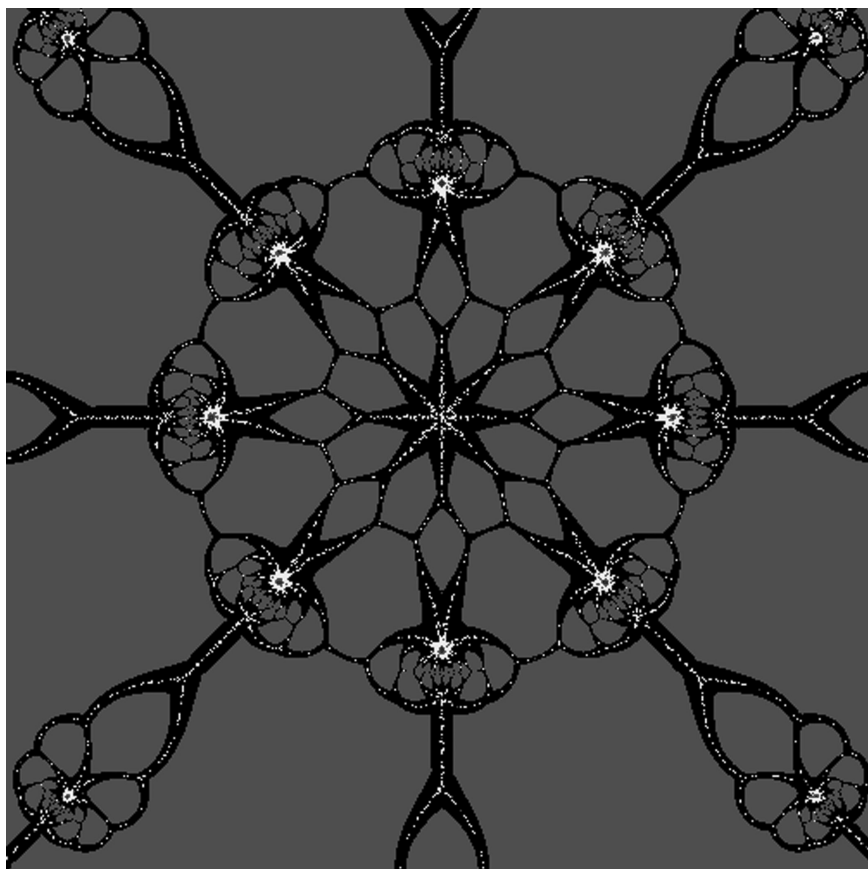


Fig. 2. *Mathematics of a Heart*, a polynomiograph based on Voronoi coloring where the initial set of points were placed in the shape of a romantic heart. (© Bahman Kalantari)

Fig. 3. *Acrobats*, a polynomiograph based on an orderly arrangement of points input via a compact formula for the underlying polynomial equation. (© Bahman Kalantari)



power. The interest of polynomiography for the general artist should be made clear below.

The magic of polynomiography is that this finite—possibly small—set of points, when combined with one or many iteration functions, imposes a coloring scheme on every other point. Thus, the initial set of points—the solutions of a polynomial equation—offer much more than the shape it defines. The polynomiographer's personal creativity and choice, and the great variety of iteration functions—which act as lenses through which to view a polynomial equation—amount to a powerful tool for artistic creation. Even with polynomials of small degree, artists can learn to produce interesting images on a laptop computer in a reasonable amount of time.

Theoretically, polynomiography can be considered a visual verification of the FTA. However, in polynomiography we are not interested merely in the roots of a polynomial equation, but in the way in which they relate to or influence all the other points within a particular region in the plane—for instance, a rectangular region that includes all or some of the roots. The polynomiography software makes use of these relationships to create artwork. In particular, in the context of visualization and art, we can reverse the role of the ancient root-finding problem and select the roots of the polynomial as we wish so as to create desirable designs. Thus polynomiography turns the root-finding problem into a tool of art and design.

Polynomiography may appeal to the artist through several properties: a relatively simple foundation; the ease with which one can generate polynomiographs; the ability to create images and designs of enormous complexity the likes of which have never been seen—or even imagined—before, or images reminiscent of familiar abstract art, at times impossible to identify as computer generated; the fact that there is meaning and human control behind the images, unlike unpredictable or random computer-generated images; and the fact that polynomiography techniques can be acquired methodically.

A reader familiar with fractal images may be inclined to view polynomiography as an extension of the well-known method for finding and plotting the basins of attraction of the roots of polynomials. This view, however, falls far short of a sufficient or fair description of polynomiography. While some polynomiographs may turn out to be fractal images obtained via plotting of the basins of at-

traction of roots, many other interesting polynomiographs are neither fractal nor based on such colorings. Even those polynomiographs that are based on the familiar coloring of basins of attractions make intelligent use of special iteration functions capable of producing anticipated shapes, as opposed to the random or limited iterations often used to generate fractals.

The infinite class of iteration functions that I have extensively studied and used in prototype polynomiography software is called the *Basic Family* [2]. This infinite family or its individual members, which include the well-known Newton's method, have been rediscovered independently by other researchers through different means. It is a fundamental family, which admits many different forms and representations. I have previously given an account of these, including new properties [3]. The Basic Family and more advanced versions are closely related to Taylor's celebrated Theorem and the FTA [4]. I have described a simple derivation of the iteration functions of the Basic Family [5]. The fact that different researchers have come across the same iteration functions may have a simple explanation: The world of iteration functions relies on the FTA.

For a polynomiographer, iteration functions within the Basic Family need not be known explicitly. They are analogous to the lenses of a camera. A photographer need only learn to work with various lenses, not learn the physics behind their construction. Likewise, polynomiography software could mask all the mathematics underlying it.

Although a polynomiograph may turn out to be a fractal image, polynomiography is not a subset of fractals, neither as theory nor as art. Indeed, polynomiography is complementary to what is known as *fractal art* (see Mandelbrot [6,7]). The polynomiographer, when producing fractal art, is working with a restricted but well-defined class: fractal images coming from special iteration functions designed for root-finding whose properties give rise to tools of design. This method provides a basis for producing fractal art with an underlying foundation, as opposed to random fractal art. One can speak of *fractal polynomiographs*. Such a term would refer to a well-defined subset of fractal art. This method can be used to create many new sets of fractal images, and hence to broaden the horizon of fractal art. On the other hand, an important feature of polynomiography is that it gives rise to a wide range of interesting non-fractal images.

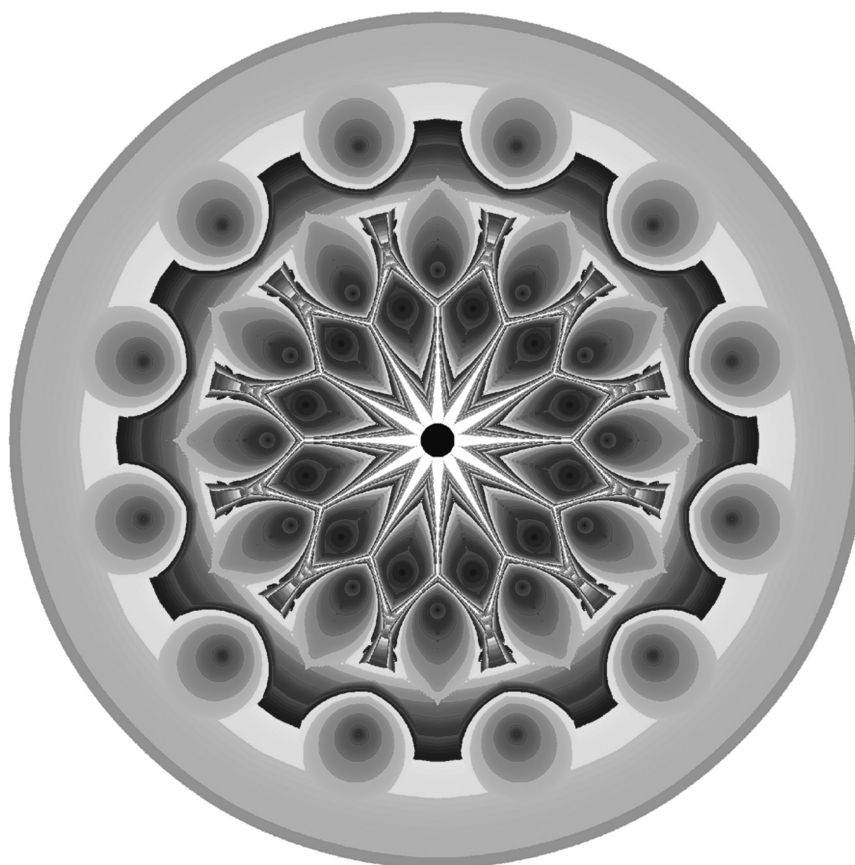


Fig. 4. A polynomiograph based on levels of convergence corresponding to a degree 36 polynomial whose design was inspired by a Persian carpet. (© Bahman Kalantari)

Visualization of polynomial root-finding methods, the best-known of which is Newton's method, was attempted before the advent of computers. In 1879, Cayley [8] questioned the convergence behavior of Newton's method for quadratic and cubic polynomials in the complex plane [9]. He was only able to analyze the convergence behavior for quadratics. For cubic equations the regions of attraction happen to form shapes known as *Julia sets*, the boundaries of which exhibit fractal behavior. The first computer visualization of this phenomenon was apparently obtained by Hubbard (see Gleick [10]). Mandelbrot's fractals [11] popularized the Julia sets and generated new interest in the computer visualizations of fractal images. Computer technology has been a significant tool not only for fractal art, but also for other forms of art inspired by mathematics and science; see, for example, Emmer [12] and Peterson [13].

While polynomiography's theoretical aspects do intersect with both the theory of fractals and dynamical systems, it has its own independent characteristics. For instance, polynomiography will not only result in a unified perspective on the theory of root-finding, but will also enable

the discovery of new properties of this ancient problem. As an art form, polynomiography is perhaps the most systematic method for the visualization of root-finding algorithms—bringing it into the realm of art and design.

POLYNOMIOGRAPHY AS A TOOL OF ART AND DESIGN

I believe that there are many similarities between photography and polynomiography. To justify this analogy, let me respond to a question often posed by viewers of my work: "Why don't you write down the underlying polynomial equation next to each of your images?"

While a defining equation might be satisfying to the viewer with respect to some polynomiographs, in many other instances a description via a single equation would be meaningless. When one is viewing a photographic portrait, the name of the person depicted is often completely immaterial. It does not give a complete or fair description of the photograph. However, I still view the suggestion as a positive reflection on polynomiography, because it means that the viewer does possess some understanding of the origins of the images, and there-

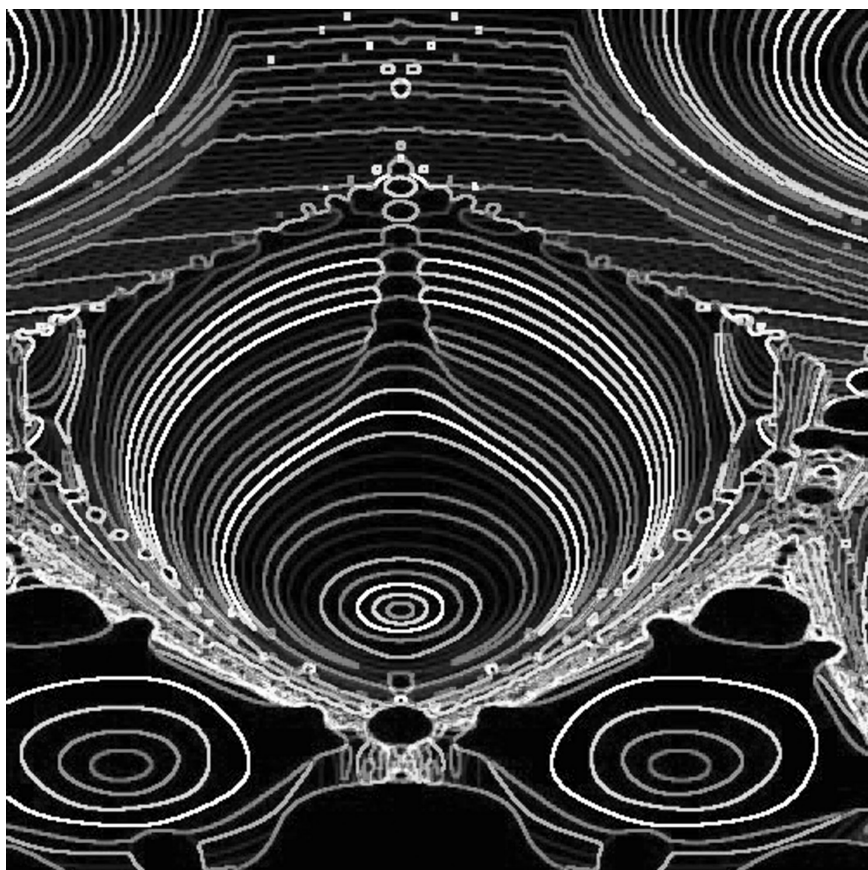


Fig. 5. *Times Square*, a polynomiograph based on levels of convergence.
(© Bahman Kalantari)

fore feels comfortable enough asking questions about the underlying equations. Contrast this case with that of typical fractal images, where the viewer often has no clue as to the source of the image. While typical fractal art does make use of iterative schemes and coloring based on them, it is seldom apparent what the iterative scheme is trying to accomplish, if anything.

In photography there are three main components: the photographer, the camera and the subject. These three components combine with other parameters to create an interesting photograph. In polynomiography, there are also three main components: the polynomiographer, the computer software that generates the polynomiographs and the underlying polynomial equation. As in photography, the final polynomiograph is produced by a combination of these three components, as well as many other parameters, such as the particular iteration function or collection of iteration functions being used, the region or area through which the polynomial equation is being viewed and the interactive coloring schemes. As in photography or painting, polynomiography

allows a great deal of creativity and choice. Here are four basic polynomiography techniques:

- Just as a photographer can shoot different pictures of a model using a variety of lenses and angles, a polynomiographer can produce different images of the same polynomial equation and make use of a variety of iteration functions, zooming approaches and interactive coloring schemes until a desirable image is discovered.
- More creatively, an initial polynomiograph, even a very ordinary one, can be turned into a desirable image based on the user's choice of coloration, individual creativity and imagination. This is analogous to carving a statue out of stone.
- The polynomiographer may employ the mathematical properties of the iteration functions, or those of the underlying polynomial, or both. This is truly a marriage of art and mathematics. The difference between this and the previous technique is that the polynomiographer can anticipate the shape of the image beforehand and thus create designs of desirable forms.

- Images can be produced as a collage of two or more polynomiographs created through one of the previous three methods. Many other techniques are possible, either through artistic compositional means or through computer-assisted design programs.

I will describe in more detail how to create two different categories of images using polynomiography. Polynomiography is based on the manipulation of a finite set of points, whether they are given explicitly or implicitly through their polynomial equation. The first category of images I will describe is based on the approximation of the *Voronoi regions* of the solutions of a polynomial equation. To visualize Voronoi regions, let us assume that we have a rectangular canvas with four random points on it marked *Blue*, *Green*, *Red* and *Yellow*. Suppose that we wish to color blue the set of all points on the canvas that are closer to *Blue* than to any of the other three colored points. The shape of this region happens to be a polygon and is called the Voronoi region of the point *Blue*. Voronoi regions of the other three points can be colored likewise.

We can think of these four points as the roots of a polynomial of degree 4. Voronoi regions and another similar coloring scheme (see Glossary) can be defined for any arbitrary set of a finite number of points placed in any arbitrary shape. The Voronoi region of any one of the points, which can be thought of as the *region of attraction* of that point, depends on the position of all the other points. We can create polynomiographs based on this coloring rule. We could do the coloring very precisely or by approximation of the Voronoi regions. Let us refer to this coloring scheme as *Voronoi coloring*, whether it is done via actual paint on a canvas or on a computer screen. Here are a few specific reasons why polynomiography can be a tool of art and design:

- Voronoi coloring produces a diverse set of images with anticipated symmetry or asymmetry.
- Voronoi coloring can be achieved via special iteration functions encoded within polynomiography software.
- With polynomiography software, Voronoi coloring can be established to any precision desired: Each member of an infinite class of special iteration functions, readily selected via an index number, generates a different approximation for the Voronoi regions of the same set. As this index increases, the approximation improves.

- The boundaries of the approximate Voronoi regions obtained via polynomiography are fractal sets. Thus, polynomiography can also produce fractal images with a great deal of variety.
- An artist can input any particular set of points, either through a polynomial equation or explicitly. In particular, it is possible to input a certain arrangement of points via a compact and simple formulation. For instance, the polynomial of degree 48 considered at the beginning of the article describes 48 points via only three input coefficients. Thus, complex patterns of points can be conveniently manipulated with polynomiography software.
- Polynomiography can help create desirable and anticipated shapes without requiring knowledge of the underlying mathematical theory. The artist can produce such shapes by learning to use the geometric effects of arithmetic operations on polynomials or complex numbers.

The computer implementation of Voronoi coloring via polynomiography generates one category of polynomiographs based on the manipulation of approximate Voronoi regions using a single iteration function. Another class of polynomiographs makes use of sets of iteration functions and iterates by moving from one to the next in a *pointwise* fashion. This method does not use repeated iterations as is normally done in iterative methods. For mathematical detail see my previous work [14].

The corresponding images in this second class are not fractal and are not based on the coloring of the basins of attraction. Rather, the colorings are defined with respect to a certain proximity to roots as measured by attributes of particular members of the special class of iteration functions in polynomiography, the Basic Family. For the sake of reference, I will call this second category of images polynomiographs based on *Levels of Convergence*.

Many other control parameters can be defined with respect to each of these two general categories, but they will not be discussed here. Next I will show images based on the two categories of polynomiographs described above.

POLYNOMIOGRAPHS BASED ON VORONOI COLORING

In the first example of polynomiographs based on Voronoi coloring, by placing fewer than a dozen points in the shape

of the letter A, and with subsequent coloring, I produced *Summer* (Fig. 1). I selected these points with a click of the computer mouse.

Mathematics of a Heart (Fig. 2) is another polynomiograph created with Voronoi region approximation. The initial set of points were placed in the shape of a romantic heart, the coloring achieved using interactive features of polynomiography software and personal choice.

Figure 3, *Acrobats*, uses an orderly arrangement of points. In this example it was necessary, as well as convenient, to input the explicit formula of the underlying polynomial equation. A polynomial equation can often give a very compact description of a set of points. For instance, $x^{100} - 1 = 0$ describes 100 points equally spaced on the circumference of a circle of radius 1 unit.

POLYNOMIOGRAPHS BASED ON LEVELS OF CONVERGENCE

Consider the polynomial $cx^{12} - 1$ where c is a complex number. When $c = 1$ the roots form a dozen points on the circumference of a unit circle, placed as are the hour marks on a clock. By using different values of c we can create two effects: rotating the points and changing the radius of the circle of roots. By multiplying three polynomials of this type, thereby producing a polynomial of degree 36, I was able to create the polynomiograph in Fig. 4. It is interesting that the inspiration behind this polynomiograph was a Persian carpet. In turn, I have had this polynomiograph turned into a high-quality hand-woven Persian carpet [15]. Polynomiography can give the blueprint for carpet designs yet to be woven, designs that would not have been possible even for the most experienced of designers.

Color Plate D No. 1 is a polynomiograph whose underlying polynomial is precisely the same as the one of Fig. 4. What has resulted in the contrast between the two is the coloring and different use of iteration functions. These were purposely done to demonstrate the diversity and choice in polynomiography. This polynomiograph too has been turned into a beautiful carpet.

Like fractal images, polynomiographs give one the ability to zoom in and discover unexpected beauty and complexity that can be used to create yet different types of images. For instance, Fig. 5, *Times Square*, was created by enlarging a small portion of a polynomiograph of the category that uses levels-of-convergence

polynomiographs, then using a commercial software and its filters to accentuate certain paths or levels.

SOME EXTENSIONS OF POLYNOMIOGRAPHY

In this article I have restricted my attention to 2D images from polynomiography. The image in Color Plate D No. 2, *Brain*, was created by Lillian Schwartz, a pioneer in the field of computer art [16], using her own techniques to modify a base polynomiograph that I obtained from a polynomial in a physics article [17]. Her image is significant in several respects. Firstly, it demonstrates the fact that an accomplished artist can use polynomiography as a tool to generate new artwork in conjunction with other media. Secondly, through her art Schwartz has made it evident that viewing certain polynomiographs with 3D chromatic glasses reveals a great deal of depth and beauty. This 3D effect in turn suggests 3D architectural structures. The design of Fig. 4, as Schwartz suggested, lends itself to sculpture. In this sense polynomiography can provide an alternative means of producing mathematically inspired sculptures. Topology has been a major source for such art, for example, in the case of Möbius strips [18]. The use of polynomiography in animation offers yet another direction with numerous applications [19]. I discuss some other applications elsewhere [20].

Finally, as with automatically generated fractals [21], it is easy to produce attractive, automatically generated polynomiographs. First of all, the k -th digits of any random number, written in ordinary base ten or in binary, can be interpreted as the coefficient of x^k of a polynomial. Then, for this single polynomial alone, an infinite number of polynomiographs can be generated where several parameters and coloring schemes can be selected randomly. Elsewhere I have presented a sample polynomiography of random numbers and its potential application in cryptography [22].

Many of the above schemes will be treated in much more detail in the future, perhaps in a book. For an interactive but experimental version of a polynomiography program, as well as links to other material with more images, the reader may visit <www.polynomiography.com>.

Acknowledgments

I would like to express my deepest gratitude to Lillian Schwartz for her invaluable suggestions regarding the preparation of this article. Not only have her

encouragement and guidance been inspirational in the past few years; they have also helped me gain a deeper understanding of art. I am also grateful to three anonymous referees whose reviews resulted in many improvements. Finally, I thank James Bickford, who carefully read my manuscript and made stylistic suggestions.

References

1. S. Smale, "The Fundamental Theorem of Algebra and Complexity Theory," *Bulletin of the American Mathematical Society* **4** (1981) pp. 1–36.
2. B. Kalantari, I. Kalantari and R. Zaare-Nahandi, "A Basic Family of Iteration Functions for Polynomial Root Finding and Its Characterizations," *Journal of Computational and Applied Mathematics* **80** (1997) pp. 209–226.
3. B. Kalantari, "An Infinite Family of Bounds on Zeros of Analytic Functions and Relationship to Smale's Bound," Department of Computer Science, Rutgers University, DCS-TR-521, 2003. To appear in *Mathematics of Computation*.
4. B. Kalantari, "Generalization of Taylor's Theorem and Newton's Method via a New Family of Determinantal Interpolation Formulas and Its Applications," *Journal of Computational and Applied Mathematics* **126** (2000) pp. 287–318.
5. B. Kalantari, "On Homogeneous Linear Recurrence Relations and Approximation of Zeros of Complex Polynomials," in M.B. Nathanson, ed., *Unusual Applications in Number Theory*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science **64** (2004) pp. 125–143.
6. B.B. Mandelbrot, *The Fractal Geometry of Nature* (New York: W.F. Freeman, 1983).
7. B.B. Mandelbrot, "Fractals and Art for the Sake of Science," in M. Emmer, ed., *The Visual Mind: Art and Mathematics* (Cambridge, MA: MIT Press, 1993) pp. 11–14.
8. A. Cayley, "The Newton-Fourier Imaginary Problem," *American Journal of Mathematics* **2** (1897) p. 97.
9. H.-O. Peitgen and P.H. Richter, *The Beauty of Fractals* (New York: Springer-Verlag, 1992).
10. J. Gleick, *Chaos: Making a New Science* (New York: Penguin Books, 1988).
11. See Mandelbrot [6].
12. Emmer [7].
13. I. Peterson, *Fragments of Infinity, A Kaleidoscope of Math and Art* (New York: Wiley, 2001).
14. B. Kalantari, "Polynomiography and Applications in Art, Education, and Science," *Computers & Graphics* **28** (2004) pp. 417–430.
15. B. Kalantari, "Polynomiography in Art and Design," *Mathematics & Design* **4** (2005) pp. 305–311.
16. L.F. Schwartz with L.R. Schwartz, *The Computer Artist's Handbook* (New York: Norton, 1992).
17. B. Kalantari, "The Art in Polynomiography of Special Polynomials," in *Proceedings of ISAMA/BRIDGES* (2003) pp. 173–180.
18. C.P. Bruter, ed., *Mathematics and Art: Mathematical Visualization in Art and Education* (New York: Springer-Verlag, 2002).
19. B. Kalantari, I. Kalantari and F. Andreev, "Animation of Mathematical Concepts Using Polynomiography," *Proceedings of SIGGRAPH, Educators Program* (2004).
20. B. Kalantari, "A New Visual Art Medium: Polynomiography," *ACM SIGGRAPH Computer Graphics Quarterly* **38** (2004) pp. 21–23.
21. J. Sprott and C. Pickover, "Automatic Generation of Quadratic Map Basins," *Computers & Graphics* **19** (1995) pp. 309–313.
22. See Kalantari [14].

Glossary

basin of attraction of a root—the set of all initial points such that the corresponding iterates of an iteration function will converge to that root.

complex number—an algebraic description of the point (a,b) in the Euclidean plane, written as $a + ib$, where $i^2 = -1$.

Fundamental Theorem of Algebra (FTA)—the polynomial equation $p(x) = 0$ has n roots. Equivalently, there are n roots $r_1, \dots, r_n$ such that $p(x) = 0 = (x - r_1) \cdots (x - r_n)$ (this implies that a polynomial equation is an algebraic description of its roots, and, con-

versely, any arbitrary set of n points $r_1, \dots, r_n$ gives rise to a polynomial equation).

iteration function—a recipe that, given any approximation to a polynomial root, however coarse it may be, provides yet another approximation, thereby allowing repetition of the process. In a neighborhood (possibly small) of any root, the iterates will converge to that root.

Julia set—The boundary of a basin of attraction of a polynomial root under an iteration function (a Julia set can be defined more generally for other iterative methods).

Newton's method—the particular iteration function $N(x) = x - p(x)/p'(x)$, where $p'(x)$ is the derivative $p'(x)$.

polynomial—an expression of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, where n is a given counting number, the coefficients $a_0, \dots, a_n$ are given complex numbers, and x is a variable.

polynomial equation—the equation $p(x) = 0$.

polynomiograph—an individual polynomiography image viewed at a selected rectangular region in the Euclidean plane.

polynomiography—the computer visualization of polynomial equations under the behavior of iteration functions.

root of a polynomial—a complex number r such that $p(r) = 0$ (a solution to the polynomial equation).

Voronoi coloring—the approximation and coloring of the Voronoi regions of a set of points, whether it is done by hand or by computer.

Voronoi regions—a partition of the points in the Euclidean plane into regions based on their proximity to a given finite set S of points. Specifically, the Voronoi region of a point s in a set S is the set of all points in the plane that are closer to s than to any other point of S .

Manuscript received 11 December 2003.

Bahman Kalantari holds a Ph.D. in computer science, a master's degree in mathematics, a master's degree in operations research, and a B.S. in mathematics and physics. He has published over 50 articles in a variety of scientific journals.