

The Fundamental Theorem of Algebra for Artists

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It's simple. It's beautiful. And it doesn't get more fundamental: Every polynomial factors completely over the complex numbers. So says the fundamental theorem of algebra.

If you are like us, you've been factoring polynomials more or less since puberty. What may be less clear is why. The short answer is that polynomials are the most basic and ubiquitous functions in existence. They are used to model all manner of natural phenomena, so solving polynomial equations is a fundamental skill for understanding our world. And thanks to the fundamental theorem, solving polynomial equations comes down to factoring. However, finding the factors is often easier said than done.

For instance, start with your favorite polynomial equation, something like $x^3 - 2x^2 - 5x + 5 = -1$. Any such equation can be rewritten so that it is equal to zero; in this case we add one to each side: $x^3 - 2x^2 - 5x + 6 = 0$. The fundamental theorem says the polynomial on the left factors completely. With a bit of work we get $(x-1)(x+2)(x-3) = 0$. Since a product of numbers is zero if and only if one of those numbers is itself zero, the factored form tells us the solutions to the original equation: $x = 1$, $x = -2$, and $x = 3$.

But wait. What about $x^2 + 1$? It can't be factored, you may protest, since a square plus one cannot be zero. While this is true for *real* numbers x , this polynomial (and according to the fundamental theorem, *all* polynomials) does factor over the complex numbers. In other words, to factor $x^2 + 1$ mathematicians first had to ask:

Is there a larger domain of numbers where it could be factored? And after a long time—several centuries into the enterprise of solving polynomial equations—the complex numbers were discovered. These are numbers of the form $a + ib$, where a and b are real numbers and i is an “imaginary” number satisfying $i^2 = -1$. Complex numbers can be thought of as points in the Euclidean plane by associating the number $a + ib$ with the ordered pair (a, b) . So complex numbers turn points in the plane into numbers to which we can apply the four elementary operations of addition, subtraction, multiplication, and division. This was a profound discovery. In this case, we have $x^2 + 1 = (x + i)(x - i)$.

We will be leaving the comforts of the real number system behind, so let's agree to use z rather than x for the variable name. It's a time-honored notational convention that z represents a complex variable.

Whether over the real or complex numbers, factoring is hard. Given n complex numbers $z_1, z_2, \dots, z_n$ (think of them as points in the complex plane), it is a cinch to construct a polynomial with those roots: Just expand the product $(z - z_1)(z - z_2) \cdots (z - z_n)$. But suppose we are given a polynomial of degree n , that is, a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where the leading coefficient $a_n \neq 0$, and the coefficients a_i may be real or complex. Finding its roots, i.e., factoring it—finding those n points in the plane—can be a formidable challenge.

Mathematicians and computer scientists have devel-

oped sophisticated algorithms to find, or to at least approximate, those pesky roots. The fundamental theorem is an important ingredient in those algorithms. It guarantees that a search for the roots will not be in vain.

VISUALIZING A COMPLEX POLYNOMIAL

It's easy to graph a function f whose domain and range are both real. Use the x axis for the domain, the y axis for the range. The graph consists of all points of the form $(x, f(x))$. Easy.

It's not so easy to graph a function when the domain and range are both complex. The domain and range are each two-dimensional, so four dimensions are required to carry out the analog of the traditional graph. Bummer.

Some compromises will facilitate visualization. Here's one that's based on *modulus*. The modulus $|z|$ of a complex number $z = x + iy$ is its distance from the origin, namely $\sqrt{x^2 + y^2}$. It is the complex analog of absolute value. To make a modulus graph of a complex function p , use the x - y plane to represent the domain, so that the point (x, y) corresponds to the complex number $x + iy$. Use the vertical axis to represent the modulus of the complex number $p(x + iy)$. Since distance cannot be negative, this 3D graph never dips below the x - y plane. For example, figure 1 shows the graph we get for the complex polynomial $p(z) = z^3 - 1$. Writing $z = x + iy$, you can check that the modulus of $z^3 - 1$ is

$$\sqrt{(x^3 - 3xy^2 - 1)^2 + (3x^2y - y^3)^2}.$$

The three dimples dip down to touch the x - y plane and represent the roots of this polynomial: the three cube roots of unity. The fundamental theorem implies

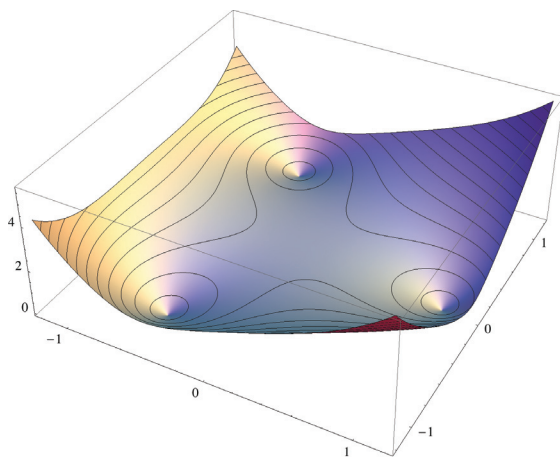


Figure 1. The 3D modulus graph of $p(z) = z^3 - 1$.

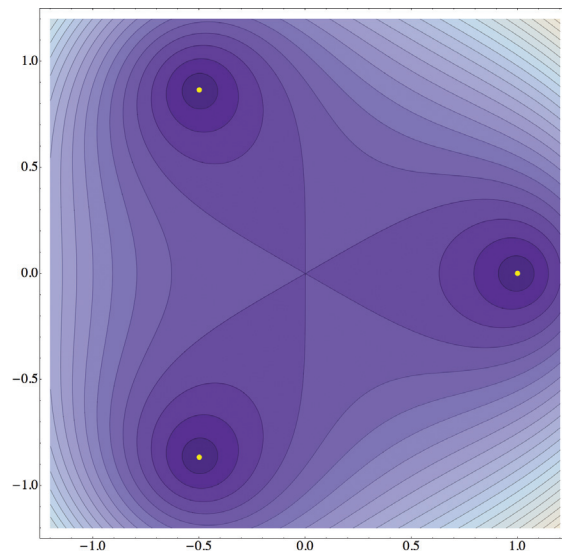


Figure 2. A contour plot of the modulus of $p(z) = z^3 - 1$.

that the modulus graph of any polynomial must dip down and touch the x - y plane.

Given the two-dimensional nature of the page you are looking at, we can improve the readability of a modulus graph by using a 2D contour plot instead. Figure 2 shows the previous graph as a contour plot, with a yellow dot at each root.

In addition to the three roots, there is another interesting point on this graph: $z = 0$ (the origin). When $z = 0$, note that $p(0) = 0^3 - 1 = -1$, which has modulus one. Look at the level curve representing all points of modulus one. If you imagine a small disk centered at the origin on the plot (as in figure 3), the level curve through the origin divides the disk into six equal sectors.

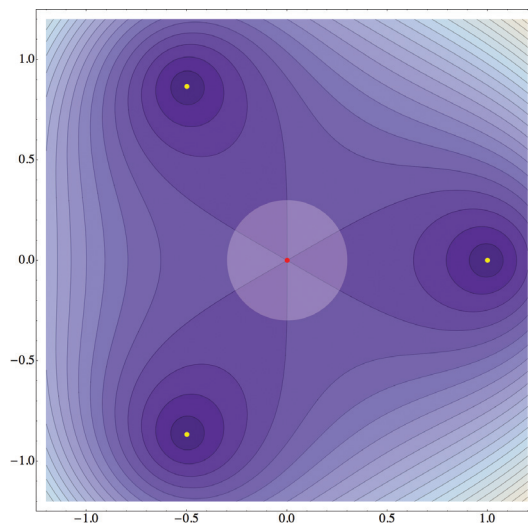


Figure 3. Examining the modulus plot in a neighborhood of the origin.

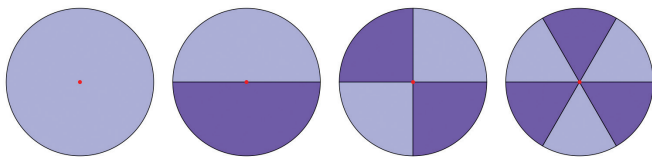


Figure 4. Disks shown at a root (left) and at points with $k = 1$, $k = 2$, and $k = 3$.

Judging by the shadings of the contour regions, three of those regions head down, toward the three roots, while the other three head up toward higher ground. Moreover, the sectors alternate between ascent and descent. This is a manifestation of a beautiful property of polynomials called the *geometric modulus principle*.

THE GEOMETRIC MODULUS PRINCIPLE

Before giving an explicit statement of the principle, we'll whet your appetite by explaining how one could predict that for the function $p(z) = z^3 - 1$, a disk centered at the origin would be divided into six ascent/descent sectors. Why six? We first take a few derivatives and evaluate them at the point in question: $z = 0$. Mercifully, differentiation in the complex landscape works just as in first-semester calculus:

$$\begin{aligned} p'(z) &= 3z^2, \\ p''(z) &= 6z, \\ p^{(3)}(z) &= 6. \end{aligned}$$

Note that $z = 0$ is a root of p' and p'' . The *third* derivative is the first that does not vanish when we plug in $z = 0$. For this example, the geometric modulus principle says that each ascent/descent sector of the disk centered at $z = 0$ has central angle $\pi/3$, where the three appears in the denominator because the *third* derivative is the first to not vanish. A central angle of $\pi/3$ produces six sectors.

More generally, suppose we are given any complex polynomial $p(z)$ of positive degree n , and a complex number z_0 . Imagine a small disk centered at z_0 in the modulus plot of $p(z)$. If z_0 is a root of p , then *every* direction from z_0 is a direction of ascent; move a small amount in any direction from z_0 and the modulus of p changes from 0 to a positive value. On the other hand, if z_0 is not a root of p , consider the derivatives

$$p'(z_0), p''(z_0), p^{(3)}(z_0), \dots$$

and let k be the position of the first nonzero number

in this list. Note that since p is a polynomial of degree n , the n th derivative must be a nonzero constant, so $1 \leq k \leq n$. The geometric modulus principle says that a small disk centered at z_0 will be divided into equal alternating sectors of ascending and descending modulus, where each sector has central angle π/k .

Looking again at the modulus plot of $z^3 - 1$, if one chooses *any* point z_0 other than 0 or one of the three roots, then $p'(z_0) \neq 0$ and so $k = 1$. A small disk centered at z_0 will be split precisely in half. The split occurs along the tangent line to the level curve through z_0 ; there is nothing particularly special here.

The interesting part concerns those points where $k > 1$, the *critical points* of p . That disks centered at these points should be *evenly* divided into sectors of ascending and descending modulus is both surprising and beautiful. See figure 4. A quick survey of the functions one encounters in a multivariable calculus course—real-valued functions of two real variables—will reveal that typical saddle points do not have equal sectors of ascent and descent. The modulus functions for complex polynomials are quite special in this sense.

THE FUNDAMENTAL THEOREM

There are numerous proofs of the fundamental theorem of algebra, and several make use of the modulus function for a polynomial $p(z)$. The basic idea behind such proofs is to first note that when z is sufficiently far from the origin, $|p(z)|$ must be large. On the other hand, within the confines of a large closed disk centered at the origin, $|p(z)|$ must attain both a minimum and a maximum. Because the values get large as one ap-

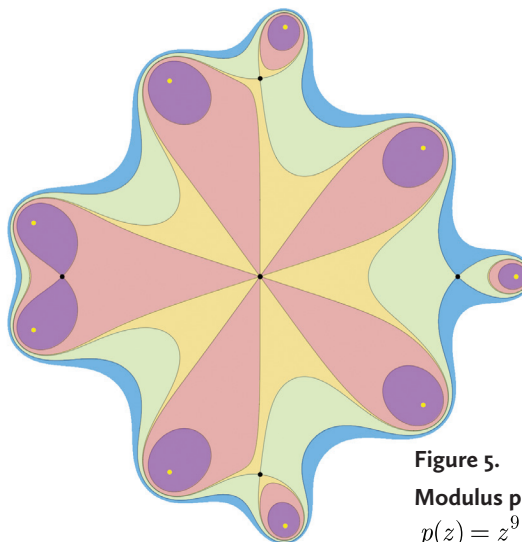
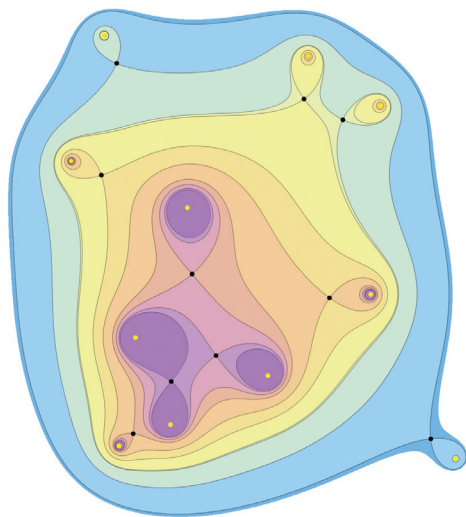


Figure 5. Modulus plot of $p(z) = z^9 - z^5 - 1$.

Figure 6.
Modulus plot
of a random
polynomial of
degree 11.



proaches the edge of the disk, all minima must occur in its interior, and so there must be a global minimum.

The proofs then argue that for any point z_0 that is not a root of p , there is a direction of descent for the modulus function, and so that point cannot be a minimum. There is then only one possibility for a minimum of the modulus function: It must be a root of p . The last step is to note that once a single root of p has been found, say z_1 , it can be factored out: $p(z) = (z - z_1)q(z)$ for some polynomial $q(z)$. But q then has a root by the same reasoning, and in a few quick steps we have factored p completely.

This line of proof can be harnessed to easily prove the fundamental theorem of algebra once the geometric modulus principle is established. In other words, the fundamental theorem is a simple logical consequence of the geometric modulus principle. Other deep results can also be derived as a consequence, such as the Gauss-Lucas theorem, and the maximum modulus principle for polynomials. It is not a stretch to characterize the geometric modulus principle as something fundamental in its own right.

FUNDAMENTAL ART

The geometric modulus principle provides a striking visual facet to the inherently abstract realm of complex polynomials. Given any collection of n complex numbers, make a modulus plot for a polynomial with roots at those numbers. If level curves—curves of constant modulus—passing through the critical points are added, the geometric modulus principle comes to life.

In figures 5 and 6, roots are shown as yellow dots,

while critical points (that are not also roots) are black dots. Each image is the modulus plot for a polynomial, showing only those level curves that pass through critical points. The geometric modulus principle guarantees that small disks centered at the critical points are always divided equally into sectors by those level curves. And of course, level curves representing distinct moduli can never touch one another.

You can also take a collection of points in the complex plane and center a disk at each point. For each disk, divide it into some even number of sectors of equal size (rotated any way you like), and alternately label each sector “ascent” or “descent.” See figure 7. There is guaranteed to be a polynomial whose modulus plot matches all the ascent/descent sectors at the indicated points! It may have additional roots and critical points, but it’s guaranteed to fit your choices precisely. ■

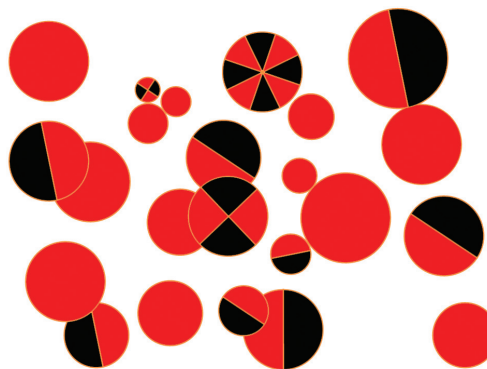


Figure 7. There is a polynomial whose modulus plot conforms to the ascent and descent sectors of these disks!

FURTHER READING

The geometric modulus principle was introduced and proved in the first author’s paper, “A geometric modulus principle for polynomials,” which appeared in the *American Mathematical Monthly* 118 (2011); 931–935.

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