POLYNOMIOGRAPHY & ART

High School Students at Rutgers

Girls Plus Math Camp at Western Illinois University, 6-8 Graders Work with Polynomiography.

Children Experience Polynomiography in Japan

What is computational thinking?

Computational thinking is a high level thought process that considers the world in computational terms. It begins with learning to see opportunities to compute something, and it develops to include such considerations as computational complexity, utility of approximate solutions, computational resource implications of different algorithms, selection of appropriate data structures and the ease of coding, maintaining, and using the resulting program. Computational thinking is applicable across disciplinary domains because it takes place at a level of abstraction where similarities and differences can be seen in terms of the computational strategies available. A person skilled in computational thinking is able to harness the power of computing to gain insights. At its best, computational thinking is multi-disciplinary and cross-disciplinary thinking with an emphasis on the benefits of computational strategies to augment human insights. Computational thinking is a way of looking at the world in terms of how information can be generated, related, analyzed, represented, and shared.

The International Society for Technology in Education (ISTE) and the Computer Science Teachers Association (CSTA) have collaborated with leaders from higher education, industry, and K–12 education to develop an operational definition of computational thinking.

Computational thinking (CT) is a problem-solving process that includes (but is not limited to) the following characteristics:

- Formulating problems in a way that enables us to use a computer and other tools to help solve them.
- Logically organizing and analyzing data
- Representing data through abstractions such as models and simulations
- Automating solutions through algorithmic thinking (a series of ordered steps)
- Identifying, analyzing, and implementing possible solutions with the goal of achieving the most efficient and effective combination of steps and resources
- Generalizing and transferring this problem solving process to a wide variety of problems

These skills are supported and enhanced by a number of dispositions or attitudes that are essential dimensions of CT. These dispositions or attitudes include:

- Confidence in dealing with complexity
- Persistence in working with difficult problems
- Tolerance for ambiguity
- The ability to deal with open ended problems
- The ability to communicate and work with others to achieve a common goal or solution.

Computational thinking in this module: POLYNOMIOGRAPHY & ART

- What is Polynomiography? In this module we will consider one of the most familiar problems to all of mankind: How to solve for the unknown? In our case the unknown is the solution to an equation, from linear and quadratic equations we all learn in middle and high school, to solving equations that scientists or mathematicians are concerned with. However, we will combine all this with a visualization called *polynomiography* that makes the whole process fun and will enable us to interact, in an easy fashion, with a polnomiography demo software to get beautiful images. In the process we will learn how to go beyond solving linear and quadratic equations, where there is no closed formula such as the quadratic formula. In fact, even the quadratic formula is not always useful when we need to approximate the solutions as decimal numbers. For instance, what is square-root of 2? Polynomiography will allow us to thinks of quadratic polynomials and general polynomials quite differently, appreciating them much more. We will learn to solve them iteratively, and also visually and to think of them as tools that not only can teach us mathematical concepts, but object that we can have fun with and generate beautiful or interesting art.

- How is the module organized? The module is divided into a 5-Day project where students learn about equations and analytic and iterative methods for solving them, going beyond the ordinary numbers and realm of complex numbers where the fun begins because we can visualize polynomials beyond the ordinay graphs, in a colorful manner. The content of each Day can be adjusted for coverage in different classes.

- Students will learn to analyze the problem of solving equations and think of different ways to solve these problems: by formulas, by iterative methods using a computer, and by a computer software that helps visualize equations and or solve them. We will think outside the box. We learn to use equations to make art, a very different application of polynomials that we were ever taught.

- Polynomial equations can be considered as abstract mathematical expressions or as ways to encrypt numbers and points (complex numbers). We can go from an equation to its solutions and the visualization (polynomiography) of the process of finding the solutions. Conversely, we can construct polynomial equations from points in the Euclidean plane, shaped as desired, followed by their polynomiography. We will get a different sense of equations.

- We will be using polynomiography software like a photographer uses a camera. We shoot images from an equation. Then we can paint them like an artist.

- We will learn to solve an equation more than one way. We will also visualize an equation in many different ways, like different paths to go from a source to a destination.

T*ypes of classes where this module can be used:*

This module could be used in a number of different classes in grades 6-12, many of which may have the flexibility in curriculum to incorporate non-standard materials. Examples of potential target classrooms include those that discuss uses of technology. These include computer science, mathematics, any science or classes, art and business classes. The material included in the module allows the teacher to select a desirable portion to cover in any appropriate class. For example, in a class intended for 6-8th graders a teacher can still introduce polynomial equations beyond linear or quadratic. In fact the polynomiography software makes it possible for anyone, even a child, to enter numbers into the software to render images much in the same way one could take a picture with a camera.

Prerequisites:

Prerequisite is dependent upon the class. If it is used as math course it is expected that student will have completed algebra 1 and geometry.

About the VCTAL project and its computational thinking modules

The VCTAL Project is a collaboration among the Center for Discrete Mathematics and Theoretical Computer Science (DIMACS) at Rutgers University, the Consortium for Mathematics and its Applications (COMAP), Colorado State University, Hobart and William Smith Colleges, the Computer Science Teachers Association (CSTA), and a number of school districts around the country. The project is funded by the National Science Foundation (NSF) to develop a set of modules and mini-modules (or teasers) for use in high school classrooms to help cultivate a facility with computational thinking in students across different grade levels and subject areas. The modules are intended to provide 4-6 days of classroom activities on a variety of topics that bring in computing and computational thinking but are drawn from applications in everyday life. Our goal is to show broad applicability of computational thinking and uses if it that extend well beyond programming. Each module will have one or more associated mini-modules or “teaser” modules. The mini-modules are intended to give a one- or two-day introduction to a topic that is either covered in or related to the larger module. Mini-modules give teachers greater ability to include computational thinking in an already-packed curriculum and allow them to experiment in a limited way.

Overall goal of the module, Diving questions, Learning Objectives for each section, Glossary at the end

DAY 1

1. Iterations as a Form of Computational Thinking

1.1 Iterating Functions, Fixed Points, Seeds, Orbits, Basins of Attraction

With the advent of high powered digital computers, iteration has become a very useful mathematical technique for solving variety of problems, in particular solving equations. In this module we will learn how iteration can be used to approximate solutions to polynomial equations, e.g. quadratic equations and consider their visualization via polynomiography, software that visualizes polynomial equations.

We can even use iterations it to solve linear equations, even though we are all familiar with how to solve a linear equation $ax+b=0$. Let's assume we didn't know how to solve a linear equation and instead try to see if we can repeatedly compute values that approach the solution, $x=-b/a$.

If we want to check whether a particular value α is the solution we simply plug it into the formula and check if $a\alpha+b=0$. Likewise, we can check if a value α is approximate solution, i.e. if $a\alpha+b$ is nearly zero.

Here is how iteration works. You take a function $f(x)$ and an initial value for x_0. Then you evaluate the function for that specific value. You can think of the function as a machine that takes an input x_0 and outputs $f(x_0)$. Next you input the value $f(x_0)$ into this machine and wait for it to compute the new output. We can repeat this process over and over, see Figure 1.

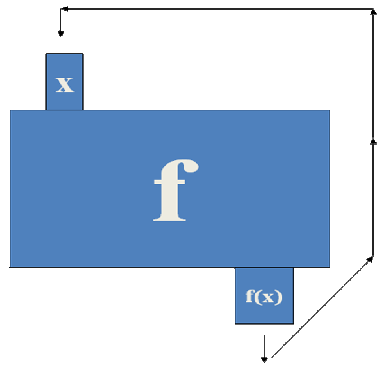


Figure 1: Input becomes output, next output becomes input.

For example, suppose $f(x) = 2x - 1$ and we start with a seed or initial value of 2. You can now create a sequence whose first term is 2 and its next term $f(2) = 2 \times 2 - 1 = 3$. The next term is found by calculating $f(3) = 2 \times 3 - 1 = 5$. The next term is $f(5) = 2 \times 5 - 1 = 9$. Continuing to do this you will create the sequence 2, 3, 7, 15, 29, 57, . . .

In what follows we will write this sequence as shown below

Example 1.

k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
x_k	**2**	**3**	**5**	**9**	**17**	**33**			

Table . Iterations of f(x)=2x-1

Notice that we will refer to the seed 2 as the 0^{th} term and 3 as the 1^{st} term or first iterate.

For a function $f(x)$ and an initial number x_0, called ***seed***, the sequence of numbers $x_0, x_1 = f(x_0), x_2 = f(x_1), x_3 = f(x_2), \ldots$, is called the ***orbit*** of x_0.

In each of the exercises below for the given function compute the orbit, starting as the given seed, given under the column 0. You may find it helpful to use a calculator or a spreadsheet if you know how to use one.

Example 2.

k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
x_k	**3**	**11**	**27**						

Table 1. Iterations of f(x)=2x+5

Example 3.

k	0	1	2	3	4	5	6	7	8
x_k	20	14							

Table 2. Iterations of f(x)=0.5x+4

Example 4.

k	0	1	2	3	4	5	6	7	8
x_k	4	9							

Table 3. Iterations of f(x)=.25x-10

For a function $f(x)$ and a given seed x_0, we say its orbit is a k-***cycle*** if
$x_0, x_1 = f(x_0), x_2 = f(x_1), x_3 = f(x_2), \ldots, x_k = f(x_0)$.

For example if we take $f(x) = x - \frac{1}{2}$, and if the seed is $x_0 = 0$, then $x_1 = 0.5$ and $x_2 = 0$. Thus we get a 2-cycle.

What is of interest to some mathematicians is this: given a function and an initial value or seed, what is the fate of the orbit. In our first example the sequence of values keeps getting larger and larger and will grow without bound. The same was true for Example 2. Something different is happening in Example 3 where the orbit appears to be approaching 8. Something similar appears to be happening in Example 4 where the orbit appears to be approaching -40/3.

Analyzing what happens when a function is iterated, it is useful to calculate what is known as the ***fixed point*** for this function.

A number α is said to be a ***fixed point*** of $f(x)$ if $f(\alpha) = \alpha$.

Referring back to Example 3, what is the fixed point for $f(x) = \frac{1}{2}x + 4$?

Solving $f(\alpha) = \alpha$ you get

$$\alpha = \frac{1}{2}\alpha + 4$$

$$\frac{1}{2}\alpha = 4$$
$$\alpha = 8$$

A fixed point α is said to be an ***attracting fixed point*** if orbits that are generated for a seed near α converge to it (i.e. get closer and closer to α.

Notice that 8 is the value that the orbit of 20 seems to be approaching. In what follows we will say that 8 is an attracting fixed point.

Going back to Example 1, let's find the fixed points of the function $f(x) = 2x - 1$. Solving $f(\alpha) = \alpha$ we get

$$\alpha = 2\alpha - 1$$
$$\alpha = 1$$

A fixed point α is said to be a ***repelling fixed point*** if orbits that are generated for a seed near α will not converge to it, no matter how close the seed may be taken (unless the seed is the fixed point itself).

A fixed point α is said to be an ***indifferent fixed point*** if it is neither attractive nor repelling.

The ***basin of attraction*** of a fixed point α is the set of all seeds whose orbit gets attracted to α.

In Example 2, the basin of attraction for the fixed point 8 is the set of all reals. Find the basin of attraction for the fixed points of the following functions.

i) $f(x) = -\frac{1}{2}x + 10$

ii) $f(x) = -x + 4$

Let us consider analyzing the orbits created by iterating $f(x) = \dfrac{x^2 + 12}{2x - 1}$. To find the fixed points we solve $\alpha = \dfrac{\alpha^2 + 12}{2\alpha - 1}$. Cross multiplying we get

$$\alpha(2\alpha - 1) = \alpha^2 + 12$$
$$2\alpha^2 - \alpha = \alpha^2 + 12$$
$$\alpha^2 - \alpha - 12 = 0$$
$$(\alpha + 4)(\alpha - 3) = 0$$

The fixed points are thus $\alpha = -4$ and $\alpha = 3$.

Use your calculator or a spreadsheet to convince yourself that

a) Both fixed points are attracting.

b) The basin of attraction for the fixed points are disjoint sets.

How to we solve a linear equation as a fixed point iteration? Suppose we want to solve $ax+b=0$. We add x to both sides to get $ax+b+x=x$. This means the solution to linear system is a fixed point of the function

$$\boxed{f(x)=a(x+1)+b.}$$

- **Informal Summary**
- A function f is a rule that move a point (number) x to point $f(x)$.
- A fixed point of f is a point that is unmoved under the rule.
- An attractive fixed point is a friendly point.
- A repulsive fixed point is an unfriendly point.
- An indifferent point is a moody point.
- An orbit is a journey that begins with an origin (seed).
- A cycle is a journey that ends up at its origin.
- Basin of attraction of a fixed point is all origins whose journey ends at the fixed point.
- Attractive fixed points have a big basin of attractions (lots of friends).
- A repulsive fixed point has isolated basin of attraction (itself and isolated friends).
- The property of attractiveness, repulsiveness, indifference is relative to f.

1.2. The Babylonian Iteration to Compute Square-Roots

The Babylonian way to approximate square roots was discovered thousands of years ago. It is an iterative method for approximating. Here is an example of how to use there method. To find a good approximation for the square root of 10 do the following.

Step 1: Make an initial guess say 3

Step 2: Using 3 as an initial value create the orbit of 3 using the function

$$f(x)=\frac{1}{2}(x+\frac{10}{x})$$

The first column in Table 1 shows the orbit of 3. Notice that the orbit very quickly converges to 3.16227766. Also, included in table 1 are the orbits for 12, -5 and 100. Notice that if the initial guess is negative, the orbit converges to the negative square root.

3	12	-5	100

3.166666667	6.416666667	-3.5	50.05
3.162280702	3.987554113	-3.178571429	25.1249001
3.16227766	3.247678535	-3.162319422	12.76145582
3.16227766	3.16340051	-3.16227766	6.772532736
3.16227766	3.162277859	-3.16227766	4.124542607
3.16227766	3.16227766	-3.16227766	3.274526934
3.16227766	3.16227766	-3.16227766	3.164201587

The Babylonian Method to Approximate $\sqrt{a}$: Pick a seed and iterate the function

$$f(x) = \frac{1}{2}(x + \frac{a}{x})$$

Activities

1. Use the Baylonian Method to find the square roots of the following numbers
   i) 27
   ii) 123
   iii) 2,435
   iv) .675
   v) -4
2. What happened in part v) of the previous exercise. Try the Babylonian Method for different negative numbers. What do you observe?

1.3. Newton's Method to Solve Quadratic Equations

In the 1700's Sir Isaac Newton devised a way to find solutions to equations of the form $f(x) = 0$. These solutions are often referred to as roots and Newton's Method uses iteration to find solutions to equations such as quadratic equation and general complex functions. Newton's Method for solving quadratic equations of the form $f(x) = ax^2 + bx + c = 0$ computes orbits of the function

$$\boxed{N(x) = x - \frac{f(x)}{f'(x)}}$$

Here $f'(x)$ is the ***derivative*** of $f(x)$ and is defined as

$$f'(x) = 2ax + b.$$

Thus

$$\boxed{N(x) = x - \frac{ax^2 + bx + c}{2ax + b} = \frac{2ax^2 + bx - ax^2 - bx - c}{2ax + b} = \frac{ax^2 - c}{2ax + b}}$$

Going back to finding solutions to $x^2+6x+8=0$, we can form the Newton Function using $a=1, b=6, c=8$. In this case ,

$$N(x)=\frac{x^2-8}{2x+6}$$

The first column in Table 1 shows the orbit of 3. Notice that the orbit very quickly converges to -2. Also, included in table 1 are the orbits for 2, -5 and -10. Notice that some orbits tend to one of the solutions and some to the other solution.

3	2	-5	-10
0.083333333	-0.4	-4.25	-6.571428571
-1.296171171	-1.507692308	-4.025	-4.925714286
-1.854628877	-1.918794607	-4.000304878	-4.22250106
-1.990774709	-1.99695048	-4.000000046	-4.02024813
-1.999957836	-1.999995364	-4	-4.000200925
-1.999999999	-2	-4	-4.00000002
-2	-2	-4	-4
-2	-2	-4	-4
-2	-2	-4	-4
-2	-2	-4	-4
-2	-2	-4	-4
-2	-2	-4	-4
-2	-2	-4	-4
-2	-2	-4	-4
-2	-2	-4	-4

Remark. If we take the polynomial $f(x)=ax^2+bx+c=x^2-2$, then Newton's method gives a formula for approximation of $\sqrt{2}$ and it gives:

$$\boxed{N(x)=\frac{x^2+2}{2x}=\frac{1}{2}(\frac{x^2}{x}+\frac{2}{x})=\frac{1}{2}(x+\frac{2}{x}).}$$

Surprisingly, this coincides with the Babylonian formula!

Activities.

1. In the above example, what is the basin of attraction for the solution x = -2?, x = -4 ?
2. Use Newton's Method to solve the following quadratics and what the basin of attraction is for each solution.

i) $x^2-1=0$ ii) $x^2+2x+4=0$

iii) $x^2-5x+6=0$ iv) $x^2-2x+6=0$

DAY 2

2. Solving Quadratic Equations

Our goal here is to solve quadratic equations via iterative methods and then enjoy them through visualization with Polynomiography software. However, first we need to develop some understanding of complex numbers and why we need to solve equations iteratively. First we will consider the solutions of a quadratic equation through the quadratic formula.

2.1. Solving Linear and Quadratic Equations Via Formulas

In middle and high schools we learn how to solve a linear equation of the form

$$\boxed{ax+b=0,}$$

where a is nonzero. The solution is

$$\boxed{x=-\frac{b}{a}}$$

In an algebra course we learn several different methods for solving quadratic equations

$$\boxed{ax^2+bx+c=0}$$

Where a is nonzero. The numbers a, b, c are real numbers, called *coefficients*. Linear and quadratic equations are examples of polynomial equations. We can solve linear equations trivially and some quadratic equations can be solved by factoring. For example, you could solve $x^2+6x+8=0$ by noticing that the sum of the roots is 6 and their product is 8 so that $x^2+6x+8=(x+2)(x+4)$ and then solving $(x+2)(x+4)=0$ which has solutions $x=-2$ and $x=-4$.

You can also solve quadratics by using the quadratic formula which states that the solutions to the quadratic equation $ax^2+bx+c=0$ are

$$\boxed{x=\frac{-b+\sqrt{b^2-4ac}}{2a}} \quad \text{or} \quad \boxed{x=\frac{-b-\sqrt{b^2-4ac}}{2a}}$$

Solving $x^2+6x+8=0$ by the quadratic formula you get

$$x=\frac{-6+\sqrt{36-32}}{2}=\frac{-6+\sqrt{4}}{2}=\frac{-6+2}{2}=-2$$

and

$$x = \frac{-6-\sqrt{36-32}}{2} = \frac{-6-\sqrt{4}}{2} = \frac{-6-2}{2} = -4$$

The discriminant of a quadratic polynomial is

$$\boxed{\Delta = b^2 - 4ac}$$

This can say a lot about the solution of a quadratic equation. If $\Delta > 0$ there are two roots. If $\Delta = 0$ there is only one solution.

$$\boxed{x = -\frac{b}{2a}}$$

If $\Delta < 0$ there are no solutions. To be exact, there are no real solutions. It turns out that there are two complex solutions. Later on, DAY 3, we will consider this case again after we have formally defined complex numbers.

The quadratic formula gives exact answers in terms of a, b, c, if $b^2 - 4ac$ is a perfect square. If for instance the coefficients are integer but the discriminant is not a perfect square, you have to use a calculator or a set of tables to find approximate values for the square root of $b^2 - 4ac$. Actually what the calculator does is to do an iteration which goes back to the Babylonians.

2.2. Solving Quadratic Equations Visually Via Polynomiography Software

You can imagine that a complex number is a point in the Euclidean plane. We will learn about polynomiography and its software later on. Here let us take the software as a way of solving a quadratic equation by means of finding the location of the roots and coloring the Euclidean plane. In the following figure (left) we have a polynomograph for $x^2 - 2 = 0$. Note that we can detect the location of the roots, square root of two and its negative. Next to it we show the polynomiography of $x^2 - 3x + 2 = 0$.

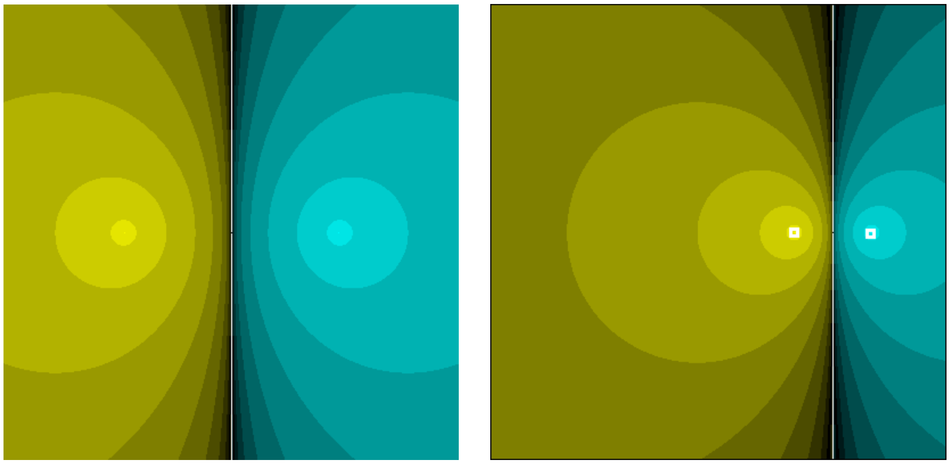


Figure 2. Polynomiograph of $x^2 - 2 = 0$. **(left)** $x^2 - 3x + 2 = 0$.

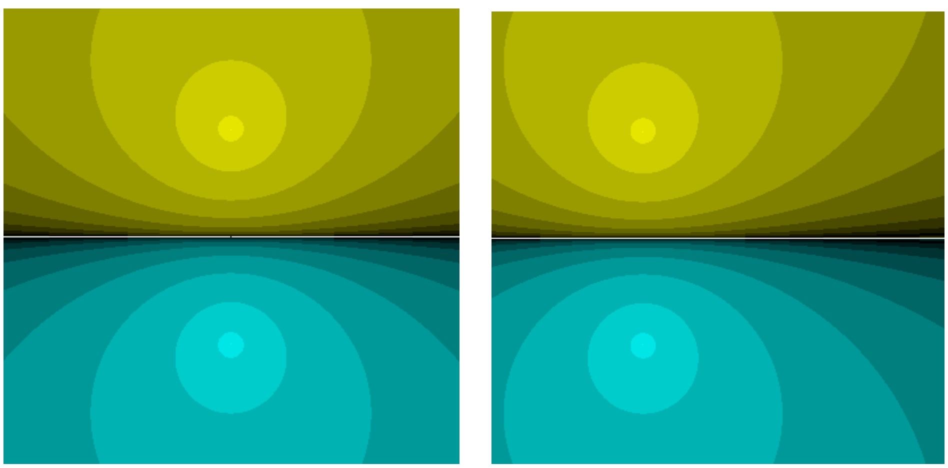


Figure 3. Polynomiograph of $x^2 + 2 = 0$. **(left)** $x^2 + 2x + 3 = 0$.

2.2. Limitations of Quadratic Formula

Consider the equation $x^2 + 6x + 8 = 0$. Whether we use the quadratic formula or by factoring, we have seen that we can find the exact solutions to this equation. But this is not the case for $x^2 - 2 = 0$ even though we know its solutions are $x = \pm\sqrt{2}$. What is the decimal form of $\sqrt{2}$? Does this number represent a geometric distance? If we take a right triangle having sides equal to one unit, by Pythagorean theorem the hypotenuse has length equal to square-root of two. So this number exists geometrically.

There are two main category of numbers, *rationals* and *irrationals*. A rational number is a positive or negative number that is the ratio of two whole numbers, e.g. 7/23, -25/17, and 14=14/1. So natural numbers (the counting numbers) are rational numbers. So are integers, the natural numbers, their negatives and the number zero. Every rational number when written in the decimal form either has a decimal expansion that has finite digits, or it has infinite digits but has a repeating pattern. The converse also holds, that is if we have a decimal expansion that is either finite or infinite but repeating, it can be written as the ratio of two whole numbers. For example

$$.25171717... = .25 + .00171717 = \frac{25}{100} + \frac{1}{100}\frac{17}{99}.$$

Obviously the right-hand-side can be combined into a single fraction. But $\sqrt{2}$ is not a rational number. The proof is not hard. Suppose $\sqrt{2} = m/n$ where m and n are whole numbers. We can assume m and n are not both even since otherwise we can factor out powers of two and simplify the ratio. Now squaring both sides we get

$$(\sqrt{2})^2 = (\frac{m}{n})^2$$

$$2 = \frac{m^2}{n^2}.$$

This implies

$$2n^2 = m^2.$$

But then m itself must be even. But this implies n is even, a contradiction. Q.E.D.

All real numbers that are not rational are irrational numbers. Any irrational number when written in decimal expansion has an infinite number of digits and has a non-repeating pattern. Since $\sqrt{2}$ is an irrational number, in order to compute its decimal expansion we would need to generate its digits. This means there is no quick way to tell what its 1000^{th} digit is. Essentially we need to calculate its digits up to that many decimal points. How do we compute its decimal expansion? Do we know if $\sqrt{2}$ is between 1 and 2? The answer is yes. We can argue this as follows: $\sqrt{2}$ is a number that when multiplied by itself results in 2. So we can say $\sqrt{2}$ lies between 1 and 2. Actually, to formally argue this we note that if u, v, w are positive numbers such that $u^2 < v^2 < w^2$, then we must have $u < v < w$. Since $\sqrt{2}$ lies between 1 and 2 we could pick the midpoint of the interval [1,2], namely 1.5, as a first approximation to $\sqrt{2}$. To get better accuracy, we can test if $\sqrt{2}$ is smaller or larger than 1.5. Since squaring 1.5 we get 2.25, we see that $\sqrt{2}$ must lie in the interval [1,1.5]. By repeatedly taking midpoints of these bracketing intervals - all of which are going to be rational numbers and hence easy to square - and comparing their squares with $\sqrt{2}$, we can continue to shrink the interval more and more. This in the process will create more and more digits with increasing accuracy in the approximation of $\sqrt{2}$. This way we get 1.5. 1.25, 1.375, etc. Actually, this approach is too slow and we will learn about a better and faster method.

Remark. One may wonder why we always consider solving a quadratic equation $ax^2 + bx + c = 0$. Why not for instance $ax^2 + bx + c = 17$? The answer is that the latter equation can always be written as a quadratic equation. In terms of notation this would be $ax^2 + bx + c - 17 = 0$. So, to use the quadratic formula we substitute for the constant term $(c-17)$ instead of c. That's why it is enough to consider solving for zeros of a quadratic equation, or a more general polynomial.

2.3. Solving Quadratic Equations Via Iterations

In what follows we will focus on quadratic equations and describe methods to solve a quadratic equation iteratively and develop the ingredients for iterative methods. It turns out that the same notions and methodology can be extended to solving a general polynomial equation.

2.3.1. Roots of a Quadratic Polynomial

Consider a quadratic polynomial

$$\boxed{p(x) = ax^2 + bx + c}, \quad a \neq 0$$

We say a number r is a ***zero*** or ***root*** or a root of $p(x)$ if :

$$\boxed{p(r) = ar^2 + br + c = 0}$$

Note that testing if a number is a root is easy because we can plug it into the equation and see if it simplifies to give zero. The reverse process is not easy, that is finding a root, or more precisely approximating a root, e.g approximation of square-root of two. By solving a quadratic equation iteratively we mean having a process that can compute a root as precisely as we wish. In order to approximate the roots of a quadratic polynomial via iterations it is necessary to define fixed points.

2.3.2. Fixed Points of a Quadratic Polynomial

The notion of root, or a zero of a quadratic polynomial is very familiar to middle and high school students. However, the notion of fixed point is not a familiar term but indeed very important when we consider iterative method for solving a quadratic equation.

Given a quadratic polynomial $p(x) = ax^2 + bx + c$ where $a \neq 0$, we say a number α is a fixed point $p(x)$ if

$$\boxed{p(\alpha) = a\alpha^2 + b\alpha + c = \alpha}$$

Remark. The fixed points of a quadratic polynomial $p(x)$ are the solutions to $p(x) = x$, equivalently $ax^2 + (b-1)x + c = 0$.

- **Summary**
- The roots and fixed points of a quadratic $p(x) = ax^2 + bx + c$ are different points.
- Its fixed points are the roots of $p(x) - x$.
- Its roots can be converted into fixed points of functions in many ways.
- For any $\lambda \neq 0$ the fixed points of $f_\lambda(x) = x - \lambda p(x)$ are the roots of $p(x)$.
- We can then apply fixed point iteration to $f_\lambda(x) = x - \lambda p(x)$ to find roots of $p(x)$.

2.3.3. Fixed Point Iterations for a Quadratic

We now define fixed point iteration. Given a quadratic polynomial $p(x) = ax^2 + bx + c$, the fixed point iteration is an iterative process to approximate a fixed point. It starts with an initial guess, called seed and for $k = 0,1,...$ define

$$\boxed{x_{k+1} = p(x_k) = ax_k^2 + bx_k + c}$$

This means that given x_0, we set $x_1 = p(x_0) = ax_0^2 + bx_0 + c$. Then $x_2 = p(x_1) = ax_1^2 + bx_1 + c$, etc. Now if the sequence converges to a number α, substituting into the iteration formula we get

$$\boxed{\alpha = p(\alpha)}.$$

Thus the fixed point iterations, if convergent, converge to a fixed point. We call the orbit (or forward orbit) of x_0 to be the sequence of points $x_0, x_1, x_2, ...$

Forward orbit looks into where each point moves under fixed point iterations. The basin of attraction of α is a set of all points whose corresponding orbit converges to α.

Problem. Recall that for the quadratic polynomial $p(x) = -0.5x^2 + x + 1$ its fixed points are $\sqrt{2}, -\sqrt{2}$. What is the basin attraction of each fixed point?

Consider a quadratic polynomial $p(x) = ax^2 + bx + c$. Set $f(x) = x - p(x)$. It is easy to show that if β is a fixed point of $f(x)$, it is a root of $p(x)$. Conversely, if α is a root of $p(x)$ then it is a fixed point of $f(x)$.

The above suggests that to find a root of a quadratic polynomial we can apply the fixed point iteration $f(x)$. Let us consider an example.

Example. Suppose we want to find the roots of $p(x) = x^2 - 2$. We take

$$f(x) = x - (x^2 - 2) = -x^2 + x + 2$$

What happens when we apply the fixed point iteration? We will see that it does not get close to $\sqrt{2}$. Let us try this. Suppose we choose $x_0 = 1$. Then $x_1 = f(x_0) = f(1) = 2$. Then

$x_2 = f(x_1) = f(2) = 0$. Then $x_3 = f(x_2) = f(0) = 2$. But this coincides with x_1. This means we will cycle indefinitely, never getting close to a fixed point. This may not happen with other seeds. For instance, try $x_0 = 3$ or $x_0 = 1.5$. You will see that except for $x_0 = \sqrt{2}$, the fixed point itself, the orbit at any other seed will not converge to the fixed point.

Recall a sequence of iterates form a k-cycle if the fixed point iteration $x_{k+1} = f(x_k)$ gives $x_0, x_1, \ldots, x_k = x_0$ but no other iterates along the cycle are identical.

Next we will try to find better fixed point iteration functions.

Consider a quadratic polynomial $p(x) = ax^2 + bx + c$. For a given non-zero number λ set $f_\lambda(x) = x - \lambda p(x)$. It is easy to show that β is a fixed point of $f_\lambda(x)$, it is a root of $p(x)$. Conversely, if α is a root of $p(x)$ then it is a fixed point of $f_\lambda(x)$.

Example. Suppose we want to find the roots of $p(x) = x^2 - 2$. We take $\lambda = 0.5$ $f_\lambda(x) = x - 0.5x^2 + 1$. What about the fixed point iteration in this case? You can check that when the seed is close to $\sqrt{2}$ the iterates do get close to it. Try this.

Why is the first example bad while the second example is good?

Given a quadratic polynomial $p(x) = ax^2 + bx + c$, we say a *fixed point* α *is an attractive* fixed point if there is some interval that contains α where for any seed in this interval the fixed point iteration will converge to it. It is a *repulsive* fixed point if no matter how close a seed is to α the iterations don't converge to α.

2.3.4. Visualization of Attractive and Repulsive Fixed Points

Let us consider the polynomiography of $f_\lambda(x) = x - \lambda(x^2 - 2)$ for some values of λ. The figures below give the behavior of the fixed point iteration and the basin of attraction of them. The fixed point $\sqrt{2}$ has a big basin of attraction, all the read area. The figures give a particular orbit as well, having connected them as a path. Note that no matter how close we choose the seed to the other fixed point, $-\sqrt{2}$, it gets attracted to $\sqrt{2}$.

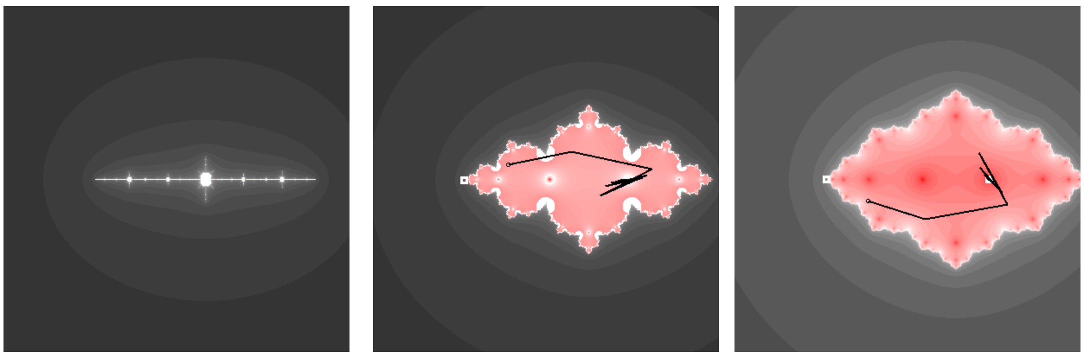


Figure 4: $\lambda = 1$ **(left),** $\lambda = .68$ **(middle),** $\lambda = .6$ **(right).**

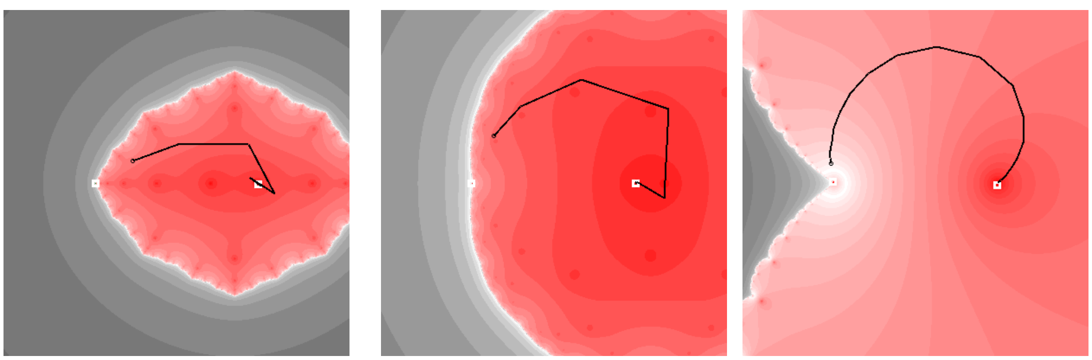


Figure 5: $\lambda = .5$ **(left),** $\lambda = .3$ **(middle),** $\lambda = .1$ **(right).**

2.3.5. Derivative of a Quadratic Polynomial (For Teahers)

NOTE: In the remaining of Section 2 the notion of derivative will be defined for a quadratic polynomial. The derivative is derived here using easy algebra that can be taught to high school students. Notion of limit concept is not used. Also the utility and significance of derivatives is discussed.

There is no easy way to determine how close to an attractive fixed point of a quadratic polynomial a seed need to be in order for the orbit to converge to it. The notion of derivative for quadratic polynomial can easily be defined and shown to be very useful in testing for attractiveness or repulsiveness of a root. Our goal here is to identify the cases when the fixed point of an iteration function for a quadratic polynomial would be attractive.

For a quadratic polynomial $p(x) = ax^2 + bx + c = 0$ its ***derivative*** is defined as

$$\boxed{p'(x) = 2ax + b}.$$

The absolute value of a number a, written as $|a|$, is a itself if it is positive and $-a$ if a is negative. Absolute value is simply the magnitude of a number. We have already defined attractive and repulsive fixed points. The derivative of a quadratic polynomial is a way to check the nature of a fixed point.

A fixed point α of $p(x)$ is attractive fixed point if $|p'(\alpha)| < 1$.

A fixed point α of $p(x)$ is repulsive fixed point if $|p'(\alpha)| > 1$.

A fixed point α of $p(x)$ is indifferent fixed point if $|p'(\alpha)| = 1$.

It can be shown that when a fixed point is attractive there is a region around it so that for any seed in that region the fixed point iteration converges to the fixed point. When a fixed point is repulsive there no region that would make the fixed point iterations converge to it. Except for choosing the fixed point as the seed itself, no matter how close we choose a seed to the fixed point, the next iteration will get away from it. An indifferent fixed point could exhibit converging behavior from one direction around the fixed point and divergence behavior at the opposite direction.

- **Summary**
- Derivative of $p(x) = ax^2 + bx + c$ is $p'(x) = 2ax + b$.
- The magnitude (absolute value) of the derivative at a fixed point α can diagnose the nature of it as attractive, repelling, or indifferent.

2.3.6. Factorization of a Quadratic Polynomial

Consider a quadratic polynomial $p(x) = ax^2 + bx + c$ and its derivative $p'(x) = 2ax + b$. Let t be any number and consider $p(t)$ and $p'(t)$. Then we have

$$\boxed{p(x) - p(t) = \frac{1}{2}(p'(x) + p'(t))(x - t)} \quad (*)$$

From (*) it follows that when $t = r$, is a root of $p(x)$, i.e. $p(r) = 0$, we get

$$\boxed{p(x) = \frac{1}{2}(p'(x) + p'(r))(x - r)}$$

Define

$$\boxed{q(x) = \frac{1}{2}(p'(x) + p'(r))}$$

Then

$$\boxed{p(x) = q(x)(x - r)}$$

This means if r is a root, then $(x - r)$ is a factor.

From (*) it follows that when $t = \alpha$, a fixed point of $p(x)$, i.e. $p(\alpha) = \alpha$, we have

$$\boxed{p(x) - \alpha = \frac{1}{2}(p'(x) + p'(\alpha))(x - \alpha)} \quad (**)$$

2.3.7. Role of Derivative of a Quadratic Polynomial in Fixed Points

Now we see that when α is an attractive fixed point then $|p'(\alpha)| = |2a\alpha + b| < 1$. Let $L = |p'(\alpha)| = |2a\alpha + b| < 1$. This means when x is sufficiently close to α then $p'(x) = 2ax + b$ is also close to $p'(\alpha) = 2a\alpha + b$ and its absolute value, L' is less than one, ($L' < 1$). But from (**) this and triangle inequality (absulte value of sum is less than or equal to sum of absolute values) we get

$$|p(x)-\alpha| = \frac{1}{2}|p'(x)+p'(\alpha)| \times |x-\alpha| <$$
$$(\frac{1}{2}|p'(x)| + \frac{1}{2}|p'(\alpha)|) \times |x-\alpha| <$$
$$\frac{L+L'}{2}|x-\alpha|.$$

Note that the average of L and L' is a number $L_0 < 1$. So from abve we have

$$|p(x)-\alpha| < L_0 |x-\alpha|.$$

This means that the fixed point iterate $x_{k+1} = p(x_k)$ when x_k is taken sufficiently close to α will satisfy

$$|x_{k+1}-\alpha| = |p(x_k)-\alpha| < L_0 |x_k-\alpha|$$

So we can say that when x_0 is sufficiently close to α then

$$|x_1-\alpha| < L_0|x_0-\alpha|$$
$$|x_2-\alpha| < L_0|x_1-\alpha| < L_0^2|x_0-\alpha|$$
$$|x_3-\alpha| < L_0|x_2-\alpha| < L_0^2|x_1-\alpha| < L_0^3|x_0-\alpha|$$

More generally,

$$|x_k-\alpha| < L_0^k|x_0-\alpha|$$

Problem. For each of the following cases - if possible - construct a quadratic polynomial $p(x) = ax^2+bx+c$ i.e. find the coefficients a, b, c such that it satisfies:

i) Both $0,1$ are fixed points.
ii) Both $0,1$ are fixed points and 0 is an attractive fixed point.
iii) Both $0,1$ are fixed points but 0 is attractive and 1 is repulsive.
iv) Both $0,1$ are fixed points, both are attractive.
v) Both $0,1$ are fixed points, both are repulsive.
vi) Both $0,1$ are fixed points but 1 is attractive and 0 is repulsive.
vii) Both $0,1$ are fixed points but 0 is attractive and 1 is repulsive.

Solution

(a): We have $p(0) = 0$. This implies $c = 0$. We also have . This implies $a+b=1$. So any polynomial $p(x) = ax^2+bx$ that satisfies $a \neq 0$, $a+b=1$ satisfies the desired condition.

(b): We want to choose $a \neq 0$, $a+b=1$, and $|p'(0)|<1$. We have $p'(x)=2ax+b$. $p'(0)=b$. So we must also have $|b|<1$.

(c): Additionally to have 1 as a repulsive fixed point we must have $|p'(1)|>1$. But $p'(1)=2a+b$. So we need to find a,b such that $a \neq 0$, $a+b=1$, $|2a+b|>1$. Substituting $a=1-b$ into the inequality we get $|2(1-b)+b|=|2-b|>1$. We can choose $b=.5$ for example. Then $a=.5$

(d): To make both attractive, we want $a \neq 0$, $|b|<1$, and $|2-b|<1$. But this is impossible.

(e): We want $a \neq 0$, $|b|>1$, and $|2-b|>1$. Choose $b=4$ for example .

Problem. Choose a quadratic polynomial $p(x)=ax^2+bx+c$ such that

(a): $\sqrt{2}$ is a fixed point, a,b,c are rational numbers.

(b): $\sqrt{2}$ is an attractive fixed point.

2.3.8. Fixed Points of a Quadratic Cannot Both Be Attractive

We have seen that for the approximation of $\sqrt{2}$ we can construct a polynomial having an attractive fixed point at $\sqrt{2}$. But the other fixed point, $\sqrt{2}$ was not attractive. We may ask if there all quadratic polynomial will have this weakness.

Theorem. Given any quadratic polynomial $p(x)$ with two distinct fixed points α_1, α_2, it is not possible to have both as attractive fixed points. In particular, the fixed point iteration cannot provide an approximation of both fixed points.

Proof. The polynomial must have the form $p(x)=a(x-\alpha_1)(x-\alpha_2)+x$. If we multiply out we get $p(x)=ax^2-ax(\alpha_1+\alpha_2)+\alpha_1\alpha_2$. So the derivative of this is $p'(x)=2ax-a(\alpha_1+\alpha_2)+1$. Now $p'(\alpha_1)=a(\alpha_1-\alpha_2)+1$ and $p'(\alpha_2)=a(\alpha_2-\alpha_1)+1$. Suppose we have $|p'(\alpha_1)|<1$ and $|p'(\alpha_2)|<1$. Then by the triangle inequality we have

$$|p'(\alpha_1)+p'(\alpha_2)| \leq |p'(\alpha_1)|+|p'(\alpha_2)|$$

But the left-hand-side is 2 while the right-hand-side is less than 2, a contradiction. Q.E.D.

2.3.9. Newton's Method Always Solves a Quadratic Equation

Newton's method for finding the zeros of a quadratic polynomial $p(x)=ax^2+bx+c$ applies the fixed point iteration to the function

$$\boxed{N(x)=x-\frac{p(x)}{p'(x)}},$$

Recall that $p'(x)=2ax+b$. Substituting for $p(x)=ax^2+bx+c$ and $p'(x)=2ax+b$ we have

$$\boxed{N(x)=x-\frac{ax^2+bx+c}{2ax+b}}$$

It is easy to check that if α is a root of $p(x)$ then it is a fixed point of Newton's function, i.e. $N(\alpha)=\alpha$. Conversely, a fixed point of Newton's function is a root of the polynomial.

The fixed point iteration under Newton's function is

$$\boxed{x_{k+1}=N(x_k)=x_k-\frac{ax_k{}^2+bx_k+c}{2ax_k+b}}$$

We claim that each root of $p(x)$ is a an attractive fixed point of $N(x)$. Note that this would be in contrast with iterations of a quadratic polynomial so that this makes Newton's iteration very useful.

Theorem. Suppose α is a root of $p(x)$. Then α is an attractive fixed point of $N(x)$. In particular, the fixed point iteration will converge to α for any seed that is close enough to α.

Proof. Recall that we have

$$p(x)=(\frac{p'(x)+p'(\alpha)}{2})(x-\alpha)$$

Thus we have

$$N(x)=x-(\frac{p'(x)+p'(\alpha)}{2p(x)})(x-\alpha)$$

Subtracting α from both sides we get

$$N(x)-\alpha=(x-\alpha)-(\frac{p'(x)+p'(\alpha)}{2p'(x)})(x-\alpha)$$

If we factor out $(x-\alpha)$ from the right hand side we get

$$(N(x)-\alpha)=(x-\alpha)(1-\frac{p'(x)+p'(\alpha)}{2p'(x)})=(x-\alpha)(\frac{2p'(x)-p'(x)-p'(\alpha)}{2p'(x)})$$

Hence

$$(N(x)-\alpha)=(x-\alpha)(\frac{p'(x)-p'(\alpha)}{2p'(x)})$$

This means when x is close to α $p'(x)$ is close to $p'(\alpha)$. Thus the fixed point iterations would be converging.

2.3.10. Basin of Attraction for Newton's Method

The basin of attraction of a fixed point of Newton's function , say α, which is a root of the quadratic polynomial $p(x)$, is the set of all points whose orbit converge to α.

Example. Consider $p(x)=x^2-2$. The basin of attraction of $\sqrt{2}$ is the entire interval $(0,\infty)$ and the basin of attraction of $-\sqrt{2}$ is the interval $(-\infty,0)$. If we were to graph this and color the basin of attractions we would get a line with two colors. Note that 0 is not the basin of attractions of either of the roots. In fact Newton's method is not defined there.

DAY 3

3. Complex Numbers, Quadratic Equations and Polynomiography

3.1. Complex Numbers

We can think of complex numbers as a mechanism to turn a location (a,b) in the Euclidean plane into a number. This is done by identifying the point with the complex number

$$\boxed{a+ib}$$

where

$$\boxed{i=\sqrt{-1}}$$

so that $i^2=-1$. With this rule we can add two complex numbers, subtract them, multiply and divide them. Examples follow.

The ***conjugate*** of a complex number $a+ib$ is the number $a-ib$ which is the mirror image of the point with respect to the x-axis.

Consider two complex numbers $1+3i$ and $2-i$. When multiplying or dividing we have to keep in mind that $i^2=-1$. Four elementary can be defined. Examples:

$$(1+3i)+(2-i)=(1+2)+(3-1)i=3+2i$$
$$(1+3i)-(2-i)=(1-2)+(3+1)i=-1+4i$$
$$(1+3i)\times(2-i)=2-i+6i-3i^2=2+5i+3=5+5i$$

Next consider division of the two numbers.

$$(\frac{(1+3i)}{(2-i)}=\frac{(1+3i)}{(2-i)}\times\frac{(2+i)}{(2+i)}=\frac{(1+3i)\times(2+i)}{(2-i)\times(2+i)}$$

Note that we multiply and divide by $(2+i)$ which is the conjugate of $(2+i)$. The denominator is simply $(2-i)\times(2+i)=4-i^2=4+1=5$. The numerator we compute by multiplying two complex number:

$$(1+3i)\times(2+i)=2+i+6i+3i^2=2+7i-3=-1+7i.$$

The ***modulus*** of a complex number $a+ib$ is written as $|a+ib|$, is given by

$$\boxed{r=|a+ib|=\sqrt{a^2+b^2}.}$$

It is easy to see

$$\boxed{r^2=(a+ib)(a-ib).}$$

3.2. Polar Coordinate

Using elementary trigonometry, given a complex number $a+ib$ we have

$$\boxed{a=r\cos\theta, \quad b=r\sin\theta.}$$

Thus we have

$$\boxed{a+ib=r\cos\theta+ir\sin\theta=re^{i\theta}.}$$

The great mathematician Euler realized the second equality above. Thus we may write

$$a+ib=re^{i\theta}.$$

This is called polar representation of a complex number and gives a geometric interpretation of the operations of multiplication and division of two complex numbers.

From Euler's formula we get the following equation that magically connects the most important of the numbers:

$$\boxed{e^{i\pi}+1=0.}$$

Suppose we have $c+id=r'e^{i\theta'}$. We see that multiplication can be done easily

$$(a+ib)\times(c+id)=re^{i\theta}\times r'e^{i\theta'}=r\times r'e^{(\theta+\theta')}.$$

Thus the product of the two complex numbers can be interpreted as a complex number whose modulus is the product of the moduli and its argument is the sum of the two arguments. In particular it is as if we rotate the first point by an angle of θ' clockwise, if θ' is positive, and counterclockwise if it is negative.

Also, we can show that

$$\frac{(a+ib)}{(c+id)} = \frac{re^{i\theta}}{r'e^{i\theta'}} = \frac{r}{r'}e^{(\theta-\theta')}.$$

It is easy to show that complex conjugates have the same modulus and the modulus of product of two complex numbers is the product of moduli of the complex numbers.

3.3. Introduction to Polynomiography

The importance of complex numbers is that everything that was said earlier about quadratics, roots, fixed points, fixed point iterations, Newton's Method, and convergence, extends to the domain of the complex numbers and we suddenly get a much more beautiful and colorful world and visual possibilities.

The term Polynomiography is a simple combination of *polynomial* and the sufffix -*graphy*. Throughout the history of mankind solving a polynomial equation has resulted in numerous significant discoveries, dating back to the Baylonians and the Summerians. The greatest mathematicians of each generation has investigated the problem for one mathematical reason or another. It has even inspired the discovery and study of the queens of computational algorithms, Newtons method which in turn is the foundation behind iterations and their visualization leading to *fractal* behavior, a term coined by Mandelbrot. In fact according to historical experts the very ideas of abstract thinking and the use of mathematical notation are largely due to the study of polynomial equations. Solving for the unknown is a problem faced by the layman on a daily basis, for instance in figuring out percentages which essentially amount to the solution of a linear equation. The computer technology and sophisticated mathematical algorithm has now resulted in a new appreciation for solving polynomial equations, going far beyond the routine fractal images.

Polynomiography could result in spectacular and diverse images, called *polynomiographs*, that not only can be appreciated as art, but induce striking appreciation of the connections between creativity in art and the intrinsic beauty of mathematics. Sample images can be found at www.polynomiography.com

A polynomial can be entered into the software either through its coefficients which is normally the way we think of a polynomial, or through its roots. For instance, if we want a polynomial to have two roots α and β, we consider

$$p(x) = (x-\alpha)\times(x-\beta)$$

The polynomiography software also allows you to select the roots by the click of the computer mouse on the screen, by selecting locations as the roots of the polynomial equation. We can select these locations as arbitrary points or selected to form a desired pattern. In the latter case the software subsequently turns the points into a polynomial equation, then an initial polynomiograph is rendered. In this sense, Polynomiography turns the problem of root-finding upside down and into a problem where the user can select the polynomial which will then undergo visualization through Polynomiography.

Furthermore, the role of algorithms become more pronounced in that it is one of the main sources of unveiling or creating beauty from such a simple looking task as solving a polynomial equation. While a dictionary definition of a polynomial equation describes it as a linear combination of an internal power of a variable such as x, a modern view of solving a polynomial equation, as well as an informal description of Polynomiography that brings it to the layman is the following (from the book, ``Polynomial Root-Finding and Polynomiography," 2008:

Solving a polynomial equation could be considered as a game of hide-and-seek with a bunch of tiny dots on a painting canvas. We hide the dots behind a polynomial equation, we then seek them using a formula or an algorithm. Polynomiography is the algorithmic visualization of the process of searching for the dots, and painting the canvas along the way.

The polynomiograph is produced via iteration functions which can serve as algorithms for approximation of polynomial roots, using a computer. Polynomiography is somewhat analogous to photography in which a photographer makes use of camera and its lenses, together with his/her personal creativity to produce an initial image, a photograph. A photographer may subsequently alter an initial photograph in order to create or enhance desired effects. Many parameters come to play to create the final image: the particular subject, the composition, the angle, the lighting, the settings, the lens, the fi
lters, and other parameters the photographer must take into account. In Polynomiography the *polynomiographer* makes use of the corresponding software and iteration functions which act as lenses, or painting brushes together with his/her personal creativity to produce an initial polynomiograph. The polynomiographer then may go through the same kind of decision making as the photographer: changing scale, isolating parts of the image, enlarging or reducing, adjusting values and color until the polynomiograph is resolved into a visually satisfying entity. Like a photographer, a polynomiographer can learn to create images that are aesthetically beautiful and individual, with or without the knowledge of mathematics or art. Like an artist and a painter, a polynomiographer can be creative in coloration and composition of images. Like a camera, or a painting brush, a Polynomiography software can be made simple enough that even a child could learn to operate.

3.3.1. Quadratic Equation with Real Coefficients and Polynomiography

Suppose we have a quadratic polynomial $p(x) = ax^2 + bx + c$, where a,b,c are real numbers. There are three possibilities

i) Two real roots, when the discriminant $\Delta = b^2 - 4ac$ is positive.
ii) One real root, when $\Delta = 0$.
iii) Two complex roots, when $\Delta < 0$ is positive. In this case the two roots are conjugate to each other.

Polynomiography of these cases are given in Figures 6, 7 and 8. Note the symmetry in these cases.

3.3.2. Connection Between Factors and Roots

A number α is a root of $p(x) = ax^2 + bx + c$ if and only if $(x-\alpha)$ is a factor, i.e.

$$\boxed{p(x) = a(x-\alpha)(x-\beta),}$$

for some number β. In fact we can determine what β would be. We know that

$$p(x) = a(x-\alpha)(x-\beta) = ax^2 - a(\alpha+\beta)x + a\alpha\beta.$$

So we have

$$a\alpha\beta = c$$

Since we know $a, \alpha,$ and c we can solve for β

$$\beta = \frac{c}{a\alpha}.$$

We can therefore see that finding roots of a polynomial is exactly equivalent to finding the factors.

3.3.3. Solutions of Quadratic Equation with Real or Complex Coefficients

Suppose we have a quadratic polynomial $p(x) = ax^2 + bx + c$, where a, b, c are real or complex numbers. What are the possible solutions in this case?

i) One real root.
ii) Two real roots.
iii) Two roots that are complex conjugates.
iv) One complex root.
v) One real root and one complex root.
vi) Two complex roots that may not be conjugate

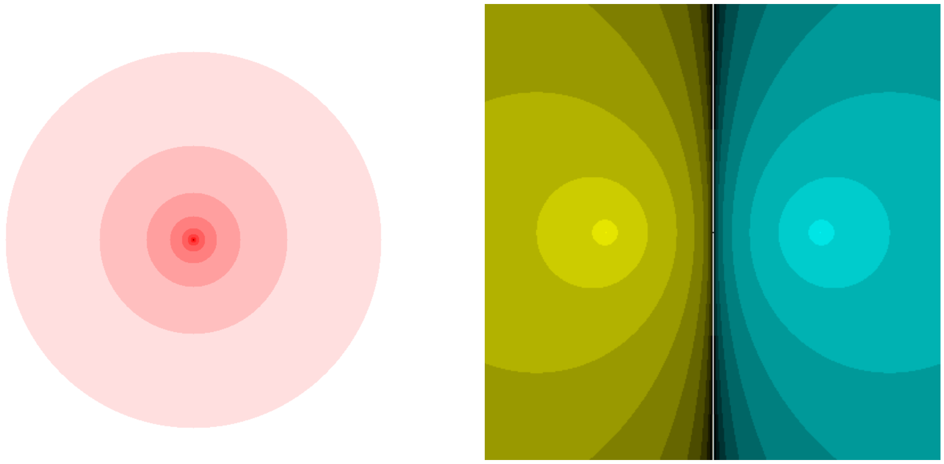


Figure 6. Polynomiograph of Case i (left), Case ii (right)

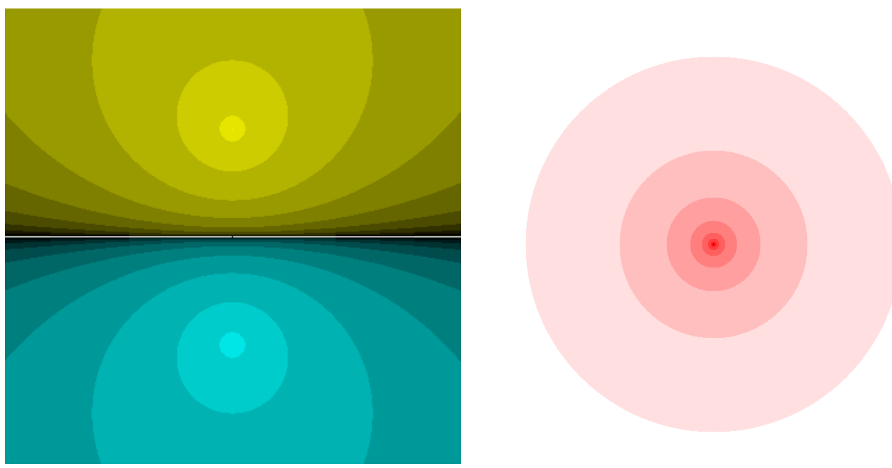


Figure 7. Polynomiograph of Case iii (left), Case iv (right)

Figure 8. Polynomiograph of Case v (left), Case vi (right)

DAY 4

4. The Fundamental Theorem of Algebra

What we have seen from the above discussion is that the quadratic formula is not always helpful in giving us a decimal approximation to the solution of a quadratic equation. However, the existence of quadratic formula inspired scholars to solve cubic equations. A cubic equation is

$$\boxed{p(x) = ax^3 + bx^2 + cx + d}, \quad a \neq 0$$

The fundamental theorem of algebra tells us that any quadratic or cubic polynomial will always have a root, no matter what we select as coefficients. This we have already seen in the case of quadratic polynomials with real coefficients. But the fundamental theorem says a root of a polynomial will always exists, no matter what the degree of the polynomial is.

It turns out that there is a formula for the solutions of a cubic equation as well. However, the formula is complicated and not easy to keep in one's head! There is also a closed formula in the case of quartic equation, an equation of the form

$$\boxed{p(x) = ax^4 + bx^3 + cx^2 + dx + e}, \quad a \neq 0$$

However, in the case quintic equations - degree 5, or higher there is a negative result which applies a very brilliant and rather complicated theory that says there is no formula that would describe the solutions in terms of radicals such as square-roots, cube-roots, etc., and defined in terms of the coefficients of the given polynomial.

What this says is that there are polynomials such that the only way to approximate their solutions is to use iterative methods. From the above we also see that if we increase the degree of the polynomial from 4 to 5 and higher it becomes cumbersome to use the letter of the alphabet so there is a more convenient way to denote a polynomial equation.

Formally, a polynomial of degree $n > 0$ is an expression of the form

$$\boxed{p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}, \quad a_n \neq 0$$

Solving a polynomial equation is to find one or more roots, i.e. solutions to

$$\boxed{p(x) = 0.}$$

When considering polynomiography of polynomials we can consider the cases when the coefficients are real, or when they are allowed to be real and complex. Of course a real number is always a complex number because the complex numbers is a superset of the reals. However, a complex number that has imaginary part is not a real number. So for instance for quadratics we can first consider those with real coefficients. We saw that there are three types: a double root, two different real roots, or a pair of conjugate roots. For a cubic polynomial with real coefficients we can have several cases such as:

i) A single real root.
ii) A double real root and another real root.
iii) A real root and a pair of complex conjugates.
iv) Three distinct real roots.

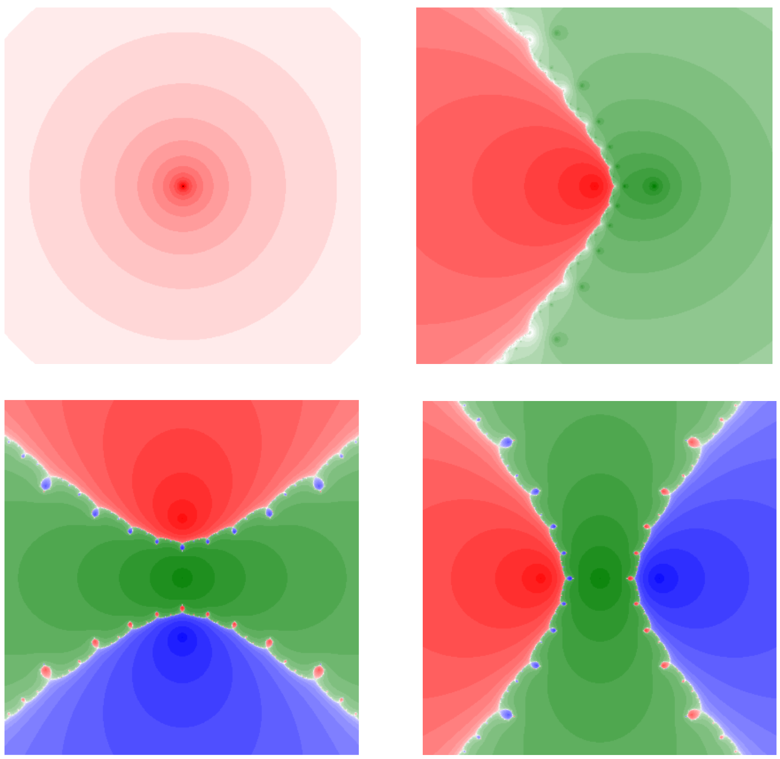


Figure 9. Case i cubic (top left), Case ii (top right), Case iii (bottom left), Case iv (bottom right)

For a quartic (degree 4) polynomial with real coefficients we can have even more cases. Some of these are:

i) A single real root.
ii) Four distinct real roots.
iii) A triple real root and a distinct second root.
iv) Two real double roots.
v) A double real root and a pair of conjugate roots.
vi) Two distinct pair of conjugate pairs.

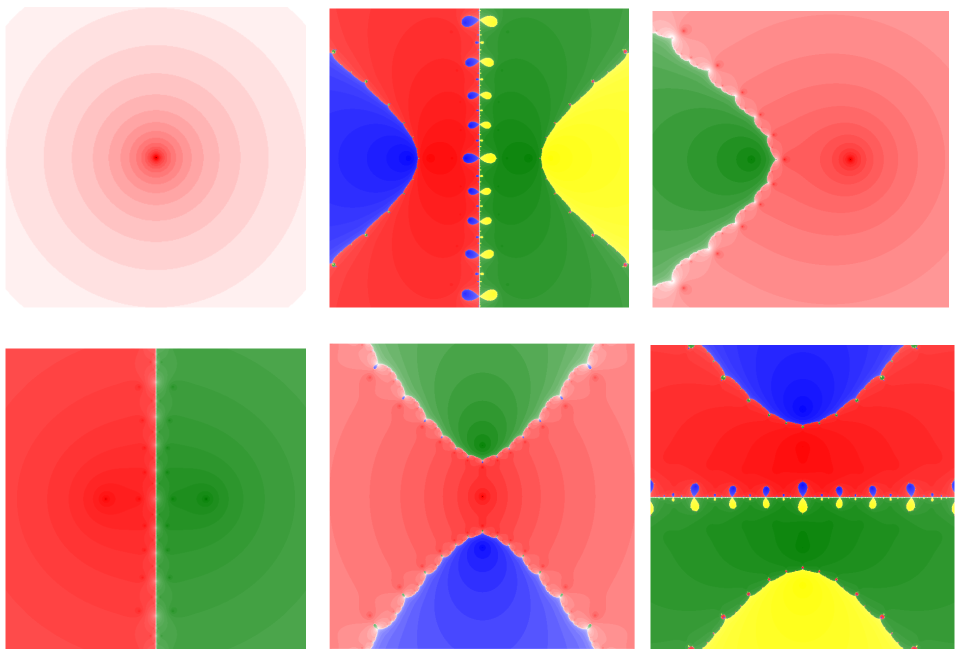


Figure 10. Polynomiography for these cases, i-iii (top row), iv-vi bottom row.

Activity: Draw as many pictures as possible for the following polynomials.

(i) $p(x) = ax^3 + bx^2 + cx + d$, consider double roots, etc.

(ii) $p(x) = x^3 - 1$, $p(x) = (x-a)^3 - 1$, different values of a, real or complex.

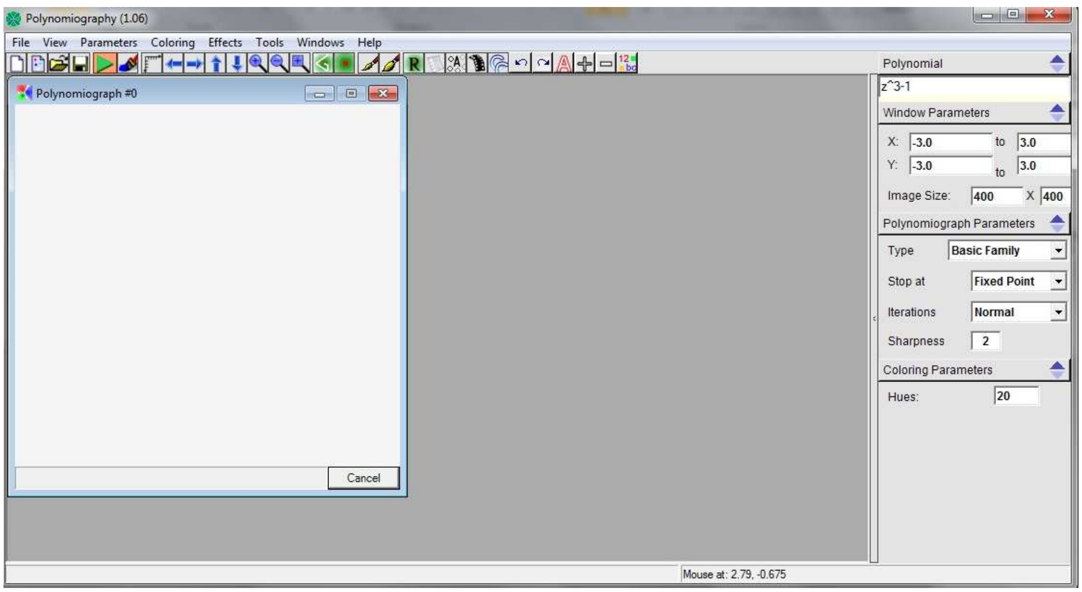
(iii) Product of polynomials such as $(x^3 - 1) \times (x^{12} - 2^{12})$.

(iv) Explore the relationship between $(x-a)$ being a factor of $p(x)$.

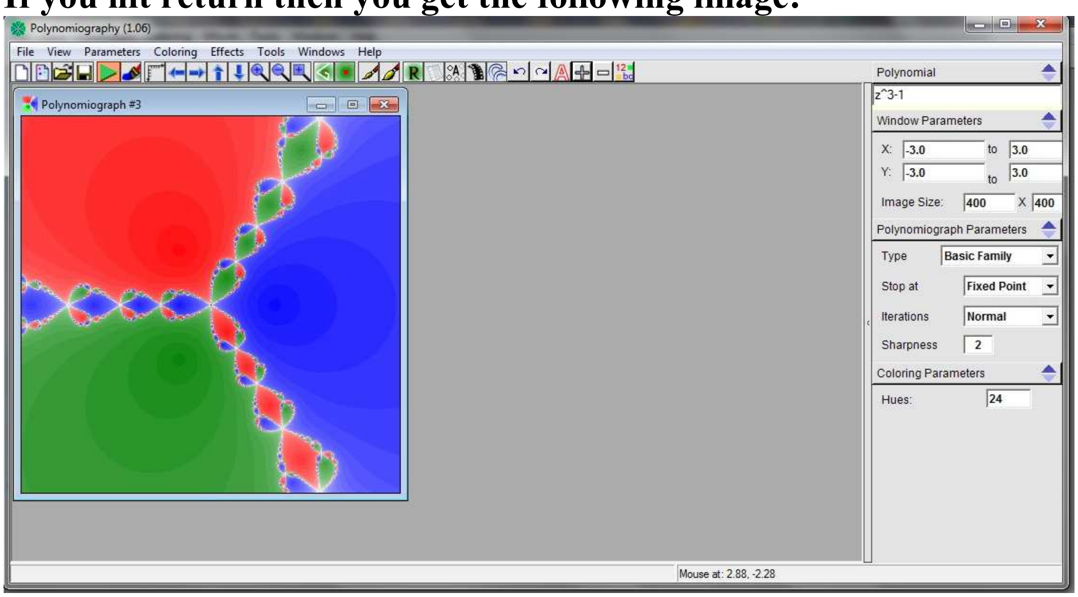
DAY 5

5. Learning to Use the Polynomiography Software

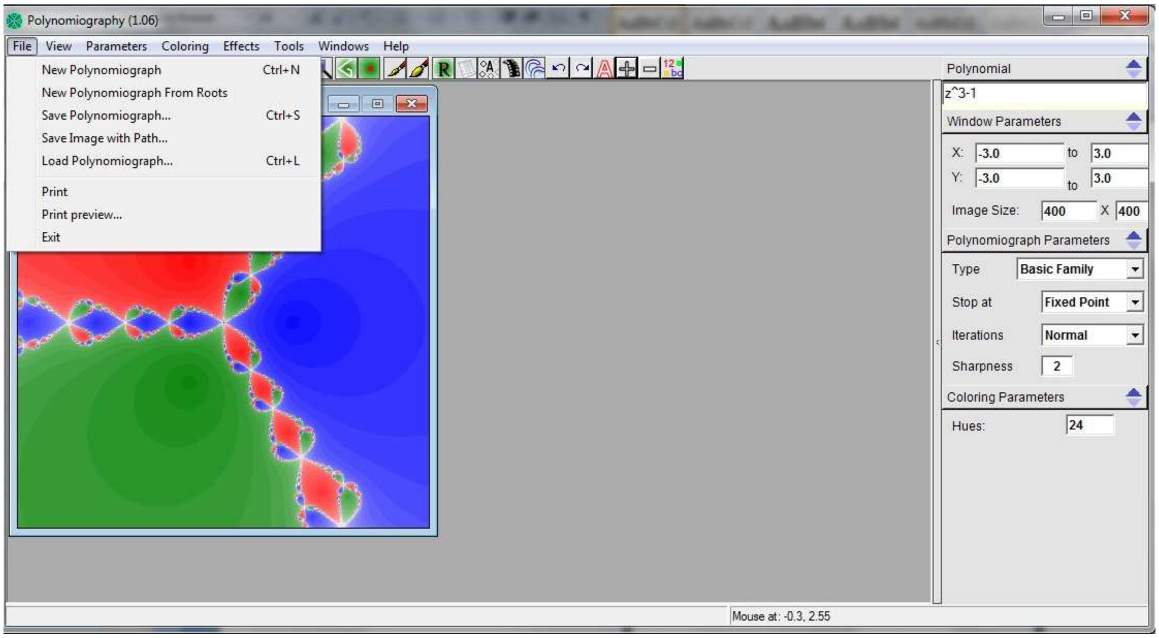
The polynomiography software gives once installed will open with a blank window such as the figure below.   Some information about how to run the software are available at the following link. Some icons have changed as the software will be improved from time to time.



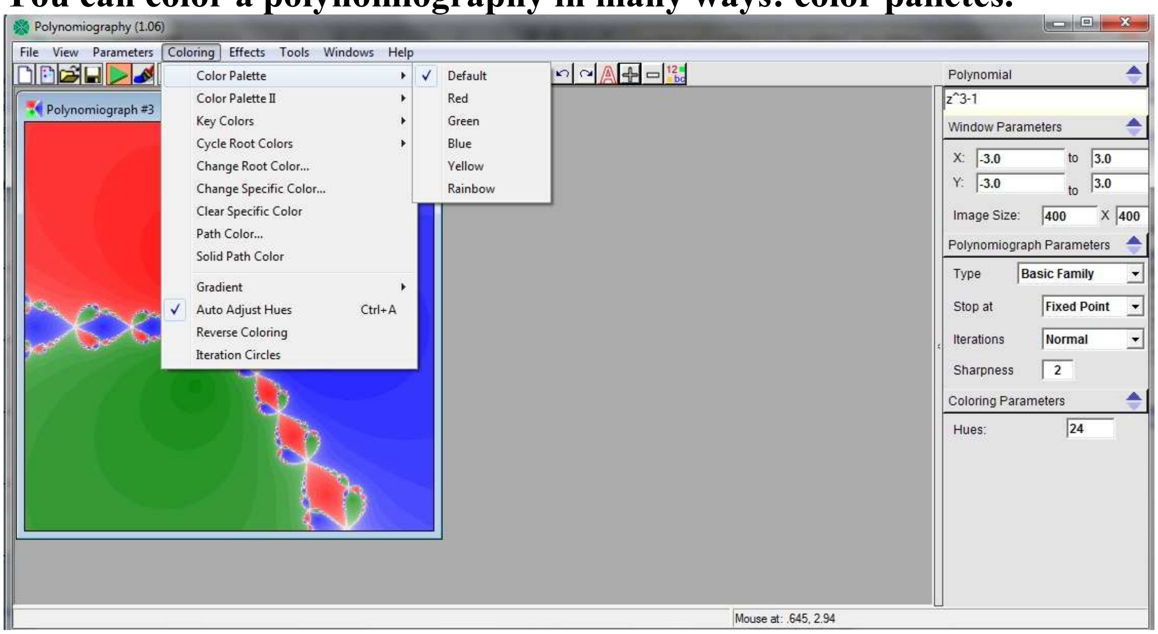
If you hit return then you get the following image:

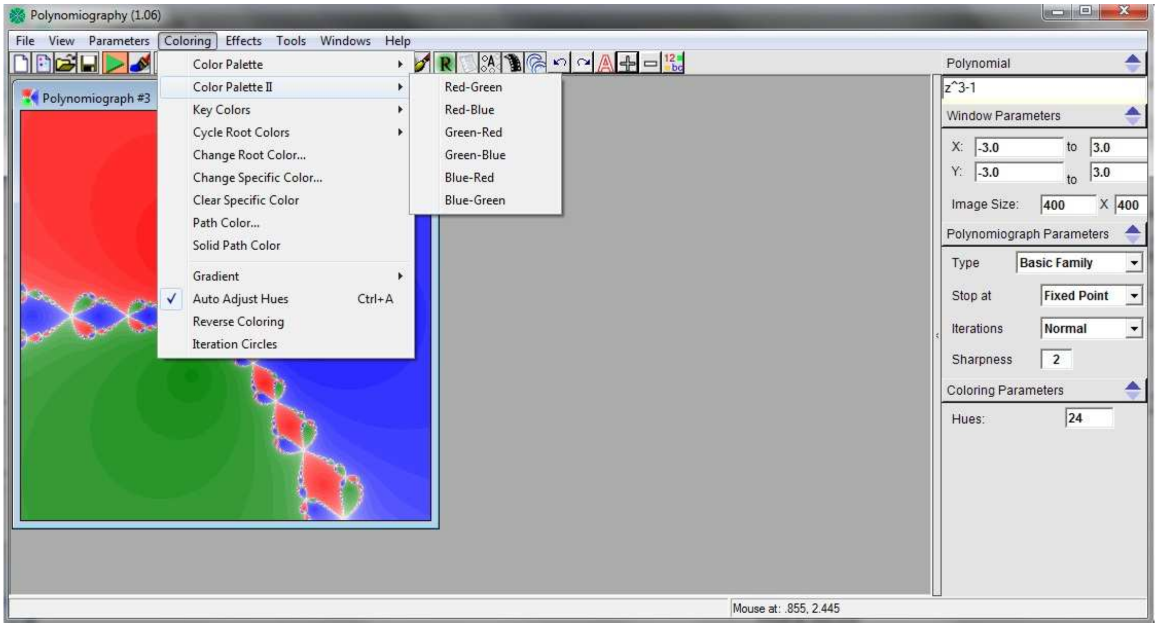


You can save an image as an png file and also an xml file to bring back to work with.

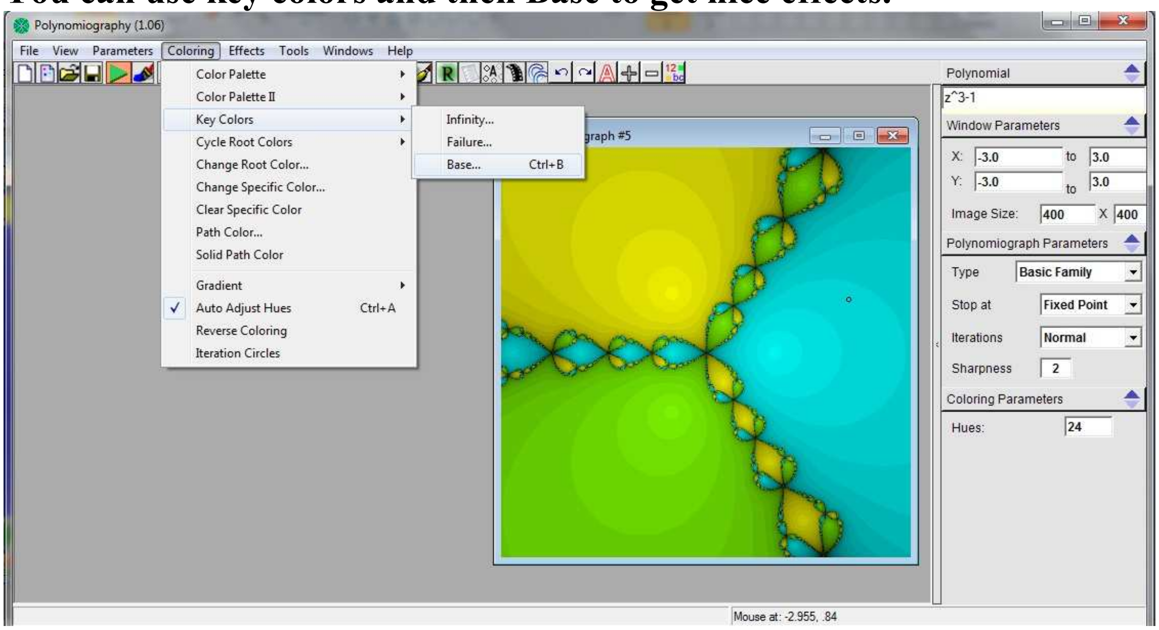


You can color a polynomiography in many ways: color palletes.

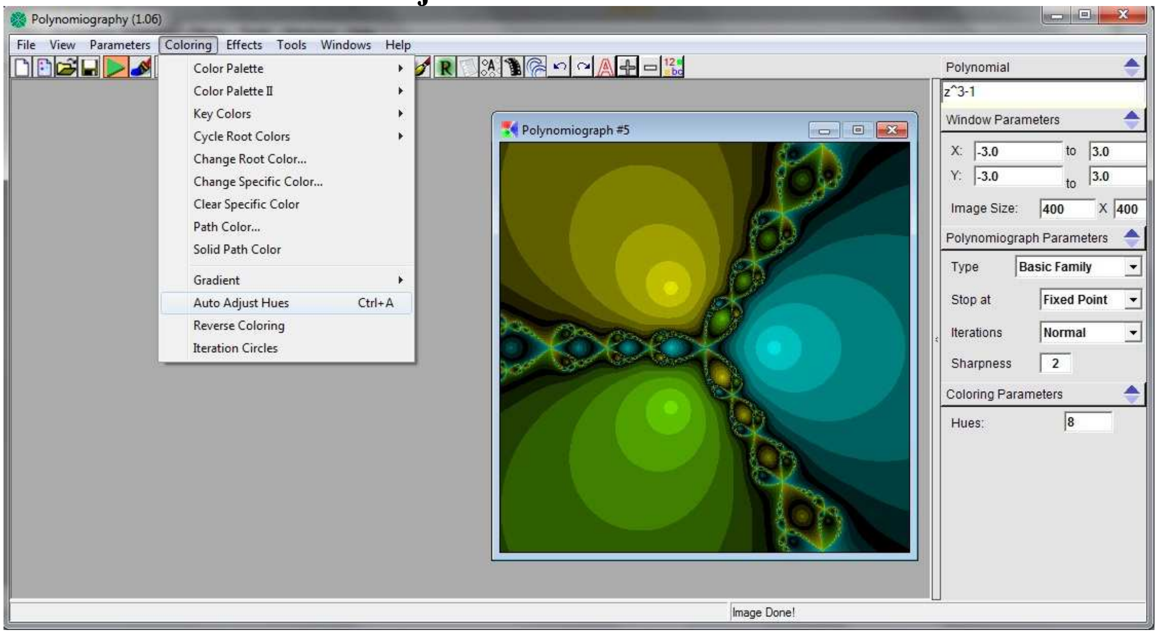




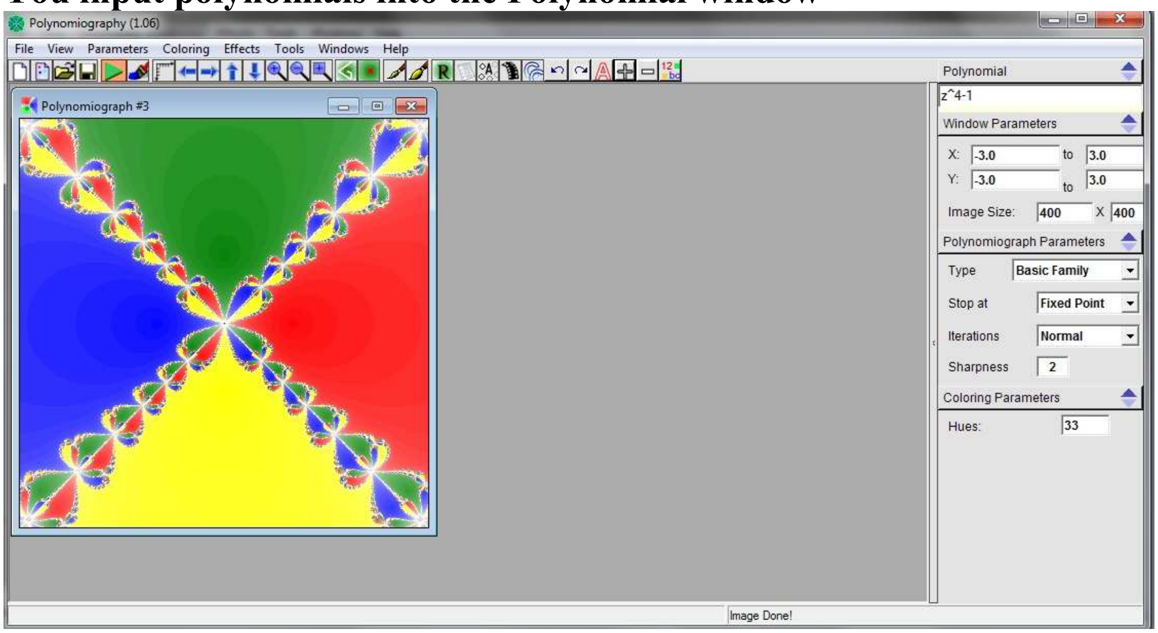
You can use key colors and then Base to get nice effects.



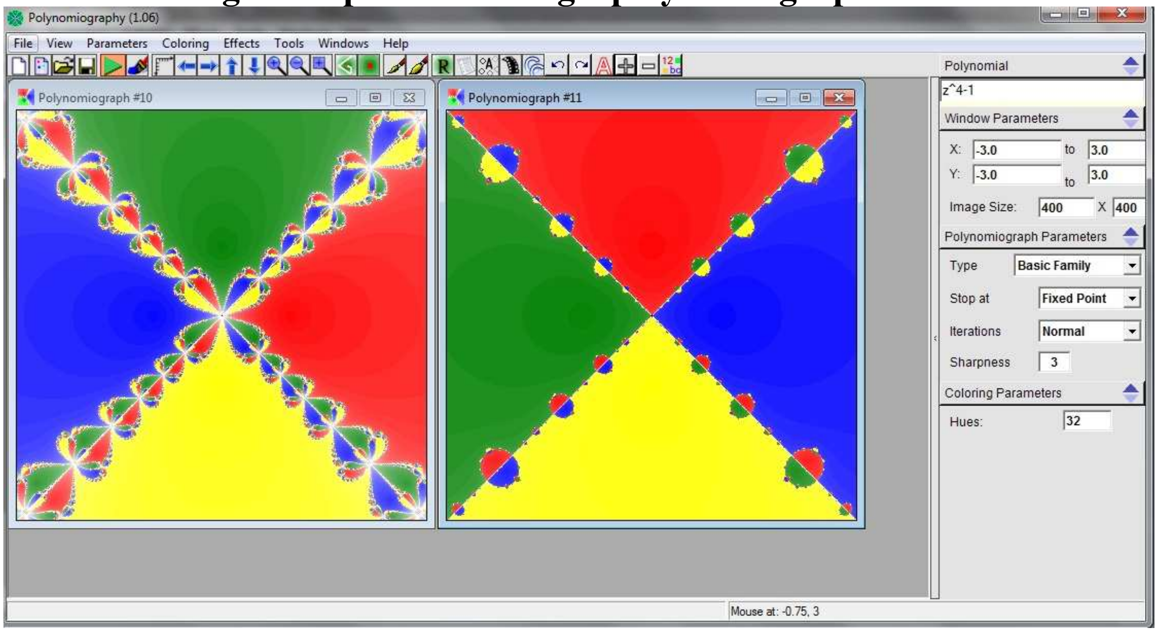
You can unmark Auto Adjust Hues and then set hues to be smaller



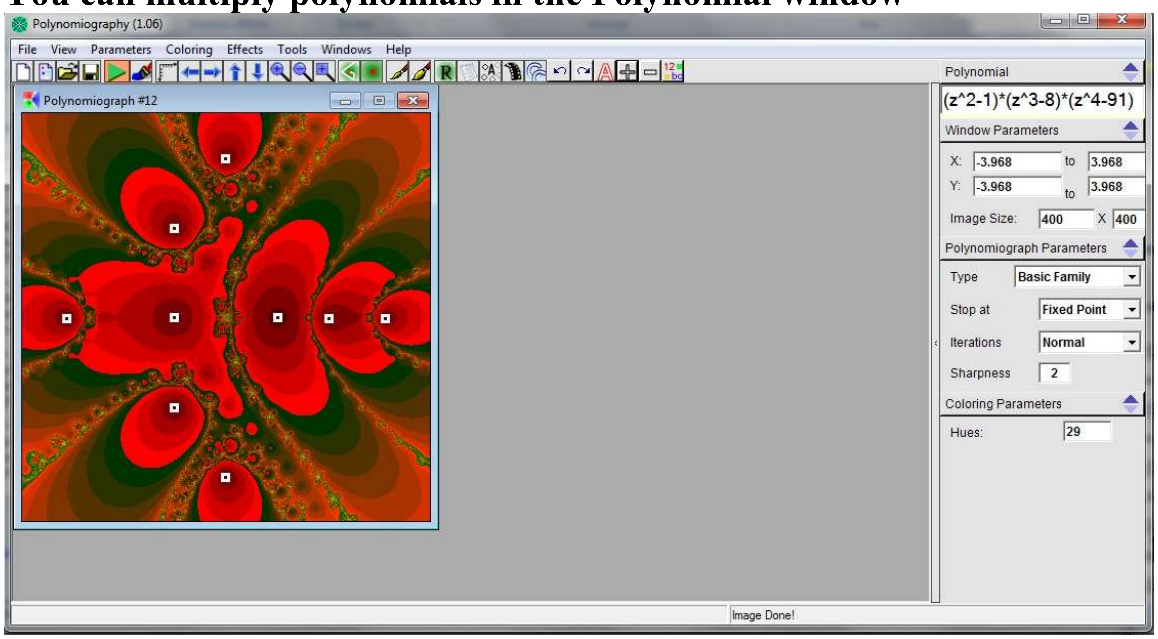
You input polynomials into the Polynomial window



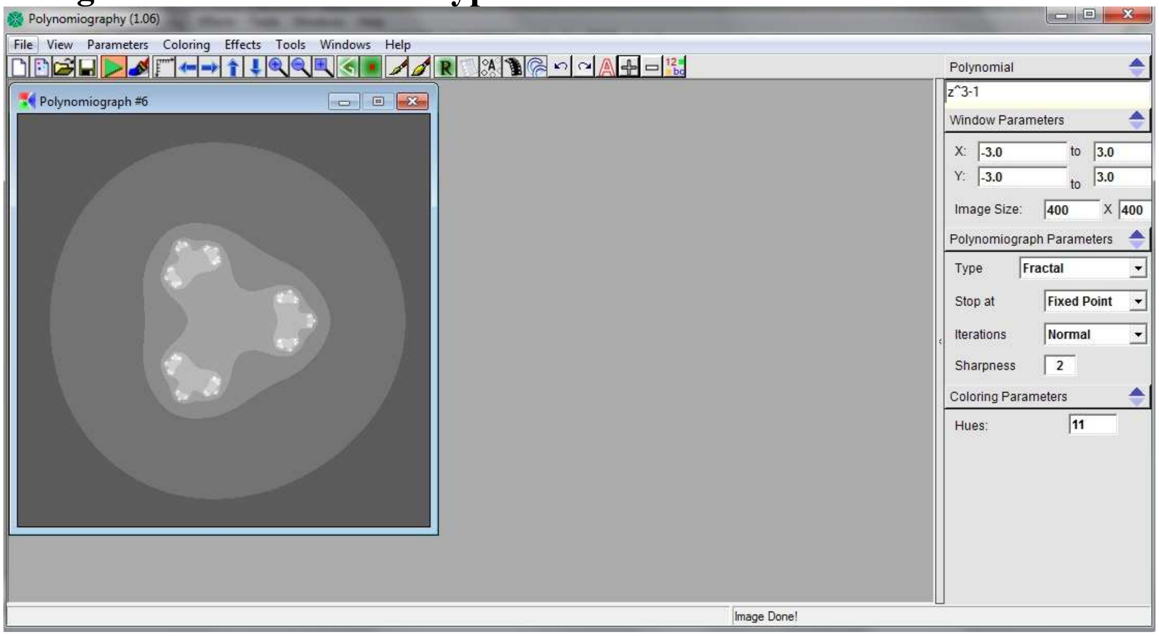
You can change Sharpness to change polynomiographs



You can multiply polynomials in the Polynomial window

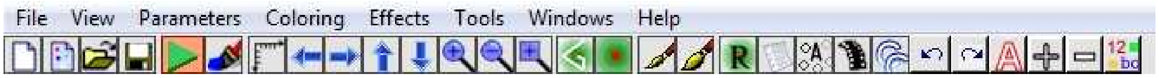


Using Fractal mode in the Type –It iterates the function in window

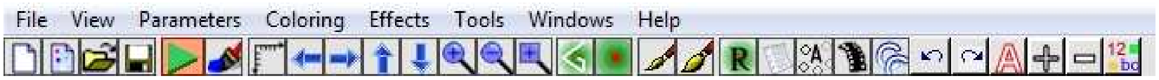


The icons are easy to get used to and to use.

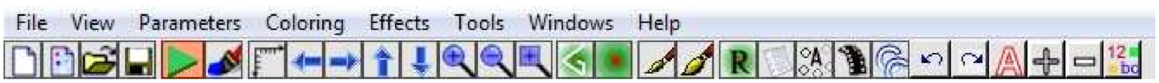
This generated a polynomiograph



This gives a blank canvas



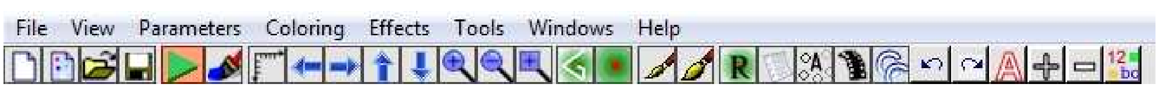
This lets you enter the roots by click of the mouse



This lets you change the color of a basin of attraction

This lets you change the color of parts

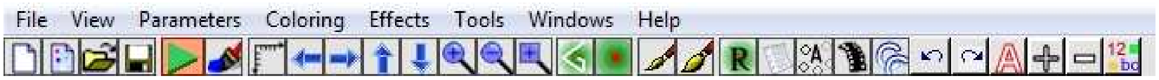
This indicates the location of the roots.



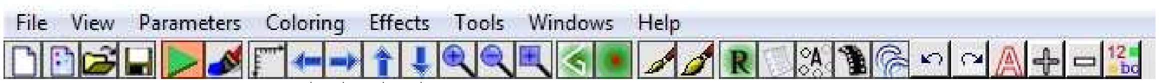
This shows the orbit as dots or a connected path.

This lets you rotate the image by entering a number.

This lets you convert a number or sentence to image.



This lets you clone a polynomiograph but have to run it.



This move the polynomiograph.

These zoom in and out.

This places the xy-coordinate lines.

These are do and undo buttons.

Activity: Draw as many pictures as possible for the following polynomials.

a. Play with the roots of unity $p(x) = x^n - 1$.
b. Start with a quadratic polynomial such as $p(x) = x^2 - 2x + 8$ and then create images of the solution of composition $p(x^n) = x^{2n} - 2x^n + 8$ for increasing values of n, say 2,3,4.
c. Use the Polynomiography software to construct polynomials by clicking points as roots and then form their polynomiographs.
d. Construct geometric shapes such as triangles and circles as roots.
e. Try to construct shapes from polynomiographs.
f. Use some of the images as a base for painting.
g. Build an activity around Polynomiography.

Glossary

Attractive Fixed Point — a fixed point such that for seeds that are close to it orbit converge to the fixed point.

Basin of Attraction of a root — the set of all initial points such that the corresponding iterates of an iteration function will converge to that root.

Complex Number — an algebraic description of the point (a,b) in the Euclidean plane, written as $a+ib$ where $i=\sqrt{-1}$ so that $i^2=-1$.

Cubic Equation — an equation of the form $p(x)=ax^3+bx^2+cx+d=0$

Cycle — a set of iterates that begin with a seed and after a finite number of iterations come back to the seed itself.

Fixed Point — of a function $f(x)$ is a number that satisfies $\alpha=f(\alpha)$.

Fixed Point of a Quadratic function— $p(x)$ is a number that satisfies $\alpha=p(\alpha)$..

Fixed Point Iteration— The process of replacing an input x_k with the $x_{k+1}=f(x_k)$.

Degree of a Polynomials $p(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ — is n.

Derivative of a Quadratic Polynomial $p(x)=ax^2+bx+c$ — is $p'(x)=ax+b$.

Derivative of a Polynomial $p(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ —is $p'(x)=na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+...+a_1$.

Fundamental Theorem of Algebra (FTA)—the polynomial equation $p(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0=0$ has n roots. Equivalently, there are n roots $r_1,...,r_n$ such that $p(x)=a_n(x-r_1)\times...\times(x-r_n)$ (this implies that a polynomial equation is an algebraic description of its roots, conversely, any arbitrary set of finite number of points gives rise to a polynomial equation whose roots are the points).

Function— a rule of assignments that given an input x computes an output f(x**).**

Indifferent Fixed Point — a fixed point such that for seeds that are close to it orbits may or may not converge to the fixed point.

Iteration Function—a recipe that, given any approximation to a polynomial root, however coarse it may be, provides yet another approximation, thereby allowing repetition of the process. In a neighborhood (possibly small) of any root, the iterates will converge to that root.

Julia set—The boundary of a basin of attraction of a polynomial root under an iteration function (a Julia set can be defined more generally for other iterative methods).

Newton's Method—the iteration of $N(x) = x - p(x)/p'(x)$, where $p'(x)$ is derivative.

Orbit— the set of iterates that begin with a seed, maybe a single point, a finite set of points (cycle), or an infinite sequence of points.

Polynomial—an expression of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ where *n* is a natural number, coefficients $a_0, a_1, ..., a_n$ are complex numbers, and *x* is a variable.

Polynomial Equation—the equation $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0$

Polynomiograph—an individual polynomiography image viewed at a selected rectangular region in the Euclidean plane.

Polynomiography—the computer visualization of polynomial equations under the behavior of iteration functions.

Quadratic Polynomial — $p(x) = ax^2 + bx + c$

Quadratic Equation— $p(x) = ax^2 + bx + c = 0$

Repulsive Fixed Point—a fixed point such that no matter how close to it a seed is chosen the orbit does not orbit converge to it.

Root of a Polynomial—a real or complex number r such that $p(r) = ar^2 + br + c = 0$ (a solution to the polynomial equation).

Seed— a number that is taken as the initial value to start a fixed point iteration.

Solving a Polynomial Equation— finding exact solutions to the equation or values that approximate the solutions.

Square-Root of a Number — a number whose square gives the original number.

Voronoi regions—a partition of the points in the Euclidea plane into regions based on their proximity to a given finite set *S* of points.

Voronoi region of a point *s* in a set *S* — is the set of all points in the plane that are closer to *s* than to any other point of *S*.