Polynomiography and Systems of Polynomials

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Vocabulary

- Monovariate: using only one variable
  - i.e. $x = 5$ is a monovariate polynomial
- Multivariate: using multiple variables
  - $xy = 5$ is a multivariable polynomial

Systems of Polynomials

$$\begin{cases} x^2 + y^2 - 1 = 0 \\ \quad x - y = 0 \end{cases}$$

This (multivariate) system has two roots! In general, it's hard to know if a system of polynomials has a finite number of roots. It's also quite unlikely given any arbitrary system

How do you tell if a System has a finite number of roots?

- Given a system of polynomials S over $R[x_1, x_2 \dots x_n]$ (ring of polynomials), S has a finite number of solutions iff the quotient ring

$$\frac{R[x_1, x_2 \dots x_n]}{S}$$

Is finitely generated!

Methods (assuming finite roots)

- Elimination Theory
  - Elimination Theory is a generalization of Gaussian Elimination
  - Seeks to express a system of polynomial equations as a system of monovariate polynomial equations
  - The problem with this is that Elimination is computationally difficult (exponential time?)
- Newton's Method (Generalized)
  - $x_{n+1} = x_n - Df(x_n)^{-1} f(x_n)$
  - $Df(x)^{-1}$ is the inverse of the Jacobian matrix
  - However, we can't solve this is the Jacobian isn't invertible. Also, calculating inverses isn't very fun

Polynomiography for Systems of Equations

- Fix all variables except for one, thus getting a monovariate polynomial which we can then do Polynomiography on
  - Then vary the variables to get an understanding of the dynamics
- Look at the norms instead of the vector values

Bounds on Roots

- There are many known bounds on the number of roots of a single monovariate polynomial
- Generalization to systems?

What I want to do

- Develop better iterative method for solving small polynomial systems
- Polynomiography for Systems
- Find some bounds on the number of roots