

1. INTRODUCTION

Definition 1. *Newton's method* is an iterative approximation method for finding roots of polynomials, although it may not always converge to a particular point. From an initial starting point z_0 , a point that is a better approximation of the closest root of a given polynomial $p(z)$ is found to be $z_1 = z_0 - p(z_0)/p'(z_0)$. The process can then be repeated such that the k^{th} iteration is $z_{k+1} = z_k - p(z_k)/p'(z_k)$.

Definition 2. *Newton's direction* for a point z_k is $-p(z_k)/p'(z_k)$.

Definition 3. The *Ellipsoid method* is an iterative method that is used in linear programming to determine whether a solution exists to a series of inequalities. First, an ellipsoid is used which is known to contain a certain number of inequalities. This ellipsoid is the collection of points such that for an $n \times n$ positive definite matrix B and a vector $z \in R^n$,

$$E(B, z) = \{x \in R^n : (x - z)^T B^{-1} (x - z) \leq 1\}$$

where z is the center, the minimum eigenvalue of B is the square of the shortest semi-axis, and the maximum eigenvalue of B is the square of the longest semi-axis. Then the ellipsoid is cut into two half-ellipsoids depending on a certain vector a where

$$HE(a) = E(B, z) \cap \{x : a^T (x - z) \leq 0\}$$

such that the half-ellipsoid that will be chosen still contains the inequalities. A new ellipsoid E' can be created which contains this half-ellipsoid, where

$$B' = \frac{n^2}{n^2 - 1} \left[B - \frac{2(Ba)(Ba)^T}{(n+1)(a^T Ba)} \right] \text{ and } z' = z - \frac{Ba}{(n+1)\sqrt{a^T Ba}}.$$

Since the new ellipsoid that contains the half-ellipsoid is guaranteed to be smaller, this process can be repeated until the center of the ellipsoid is within the region formed by the inequalities, which means there is a solution.

Theorem 1. *For any polynomial $p(z)$ and any point z in the complex plane, there will be a root of the polynomial in Newton's direction.*

2. ELLIPSOID-NEWTON METHOD

By using the methods and theorems described above, we developed a new algorithm called the Ellipsoid-Newton method that can be used as an iterative method when creating polynomiographs.

- (1) For a given polynomial $p(z)$, we consider a rectangular portion of the complex plane containing all the roots of $p(z)$. We split this up into a predetermined number of pixels. Each pixel represents a coordinate $z = c + di$, which is determined by the size of the rectangle and the number of pixels it is split into.
- (2) We find the roots of $p(z)$.
- (3) For each root, we find the pixel representing the closest coordinate to the root and assign a color to that pixel.

- (4) We then determine an $\epsilon = .89^k$, which will be an interval around a root such that if an iteration of the Ellipsoid-Newton method is within this interval, the process will stop. The constant .89 is used because each new ellipse created from the Ellipsoid method has an area less than or equal to 89% of the area of the previous ellipse in dimension 2.
- (5) For each z represented inside the rectangle, we find the point z_f in the rectangle that is the farthest distance away from z .
- (6) Let r be the distance between z and z_f and let it be the radius of a circle, with z as its center. We initialize some $k_0 = 0$, $z_0 = z$, and a matrix B to be:

$$B = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix}$$

- (8) While k_0 is less than k ,
 - (a) Find $|p(z_0)| = \sqrt{c^2 + d^2}$
 - (b) If $|p(z_0)| < \epsilon$, assign the pixel associated with z the same color as the closest root and leave the loop.
 - (c) Let $a = p(z_0)/p'(z_0)$. This will allow the new ellipse to contain all the points in Newton's direction from z_0 .
 - (d) Let $B' = \frac{4}{3}[B - \frac{2(Ba)(Ba)^T}{3(a^T Ba)}]$ and $z' = z_0 - \frac{Ba}{3\sqrt{a^T Ba}}$.
 - (e) Now, $B = B'$ and $z_0 = z'$
 - (f) Add one to k_0 .
- (9) If k_0 is equal to k , the pixel is colored white. Otherwise, the pixel is shaded such that the color is $\frac{k-k_0}{k}$ of the original brightness.